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George A. Christodoulakis
Stephen E. Satchell

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BANK OF GREECE
Economic Research Department – Special Studies Division
21, E. Venizelos Avenue
GR-102 50 Athens
Tel: +30210-320 3610
Fax: +30210-320 2432

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EXACT ELLIPTICAL DISTRIBUTIONS FOR MODELS OF CONDITIONALLY RANDOM FINANCIAL VOLATILITY

George A. Christodoulakis
Manchester Business School and Bank of Greece

Stephen E Satchell
Trinity College, University of Cambridge and Bank of Greece

ABSTRACT

Assuming the time series of random returns to be jointly elliptical, we derive a relationship between its conditional variance and the probability density function of the conditioning set. In the case that such a relationship is linear in a quadratic form for of the conditioning variables, we show that the probability density function of the conditioning variables is multivariate t. This result is then applied to models of conditionally random volatility and used to derive exact results for the GARCH(p,q) class of processes previously thought to be intractable.

Keywords: Elliptical Distributions, Financial Asset Returns, Conditional Volatility, GARCH.
JEL Classification: C22, G11, G12

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Correspondence:
George A. Christodoulakis,
Advisor to the Governor,
Bank of Greece, 21, E. Venizelos Ave.,
102 50 Athens, Greece
Tel.: +30 210-320 5006, Fax: +30 210-3205427
E-mail: gchristodoulakis@bankofgreece.gr

1. Introduction

ARCH and GARCH models are familiar examples of a large number of econometric specifications that assume independent error processes and a pattern of conditionally random volatility, see Engle (1982) for the origin and Bollerslev *et al.* (1992) for a summary. Ignoring trivial cases, there are no finite sample results known to the authors that give anything other than numerical/Monte Carlo measures for the unconditional distributions of the volatility or the variable whose volatility is under investigation, see Knight and Satchell (1998) for a discussion of the difficulties. The building of conditional volatility models is based on specific assumptions about the distribution generating the conditioning information set, usually asset returns. Likewise, the form of the asset volatility generating process is also based on specific assumptions. However, there are no theoretical results available linking the form of conditional volatility process and the form of the distribution generating asset returns.

The purpose of this paper is to provide some results in this direction. We assume that our variables are generated by distributions within the elliptical class and extend a result by Chu (1973) which enables us to characterize exactly the distribution of the variables given the pattern of conditional volatility. We then investigate the implications of our results for arbitrarily general GARCH processes. In particular, we indend to answer to questions of the following form: if asset returns are generated withing the Elliptical class of distributions and asset volatility follows a particular process, e.g. GARCH, then what should be the exact form of the asset return distribution? Elliptical distributions are of interest to economists for several reasons, they are a tractable generalization of multivariate normality, (see Genton 2004; 2005), they are used in portfolio selection and asset pricing theories (Ingersoll, 1987; Zhou, 1993; Hodgson et al, 2002 and Vorkink, 2003); and they give interesting extensions to standard micro-structure models, (see Owen and Rabinovitch, 1983 and Foster and Viswanathan,1993 for details).

Our general results which are extensions of results in Satchell (1994), are then specialized to patterns of conditional volatility which are linear functions of a quadratic form of the conditioning variables, resulting in that such an elliptical distribution will be multivariate t. Finally we further specialize to the class of GARCH(p,q) processes and derive the appropriate distribution. Our basic theoretical results are given in Section 2, followed by our results on GARCH in Section 3. We present some remarks on these results and their relationship to what we already know about GARCH models and our conclusions in Sections 4 and 5 respectively.

2. The Link Between Volatility and Elliptical Asset Returns

Following the definition of Blake and Thomas (1968), we shall define the vector $\mathbf{x}$ to be elliptically distributed if and only if its probability density function (pdf) can be expressed as a function of a quadratic form, i.e.

$$p(\mathbf{x}) = f\left(\frac{1}{2}\mathbf{x}'C^{-1}\mathbf{x}\right) = f(s) \tag{2.1}$$

where $s = \frac{1}{2}\mathbf{x}'C^{-1}\mathbf{x}$. The matrix C is positive definite and is known as the *characteristic matrix*. It is proportional to the covariance matrix if the latter exists.

There is a unique function $w(\ell)$ defined as $0 \leq \ell \leq \infty$ which together with C defines the elliptical distribution, see Theorem 1 of Chu (1973). Indeed Chu shows that for $\mathbf{x}$ a $T \times 1$ vector,

$$w(\ell) = (2\pi)^{\frac{T}{2}} \det(C)^{\frac{1}{2}} \ell^{-\frac{T}{2}} L^{-1}(f(s)) \tag{2.2}$$

where $0 \leq \ell \leq \infty$ and $L(\cdot)$ is the Laplace transform operator. Chu gives examples of $w(\ell)$ functions for some of the more familiar members of the elliptical class. There is a linkage between the pdf $p(\mathbf{x})$ and $w(\ell)$. Equation (2.2) can be inverted so that

$$p(\mathbf{x}) = \int_0^\infty w(\ell) N_x\left(\ell^{-1}C\right) d\ell \tag{2.3}$$

where $N_x(\ell^{-1}C)$ is the pdf of a $T \times 1$ vector of normal random variables with mean zero and covariance matrix $\ell^{-1}C$.

In what follows we shall restrict C to be a diagonal matrix with diagonal elements not necessarily equal. We regard $\mathbf{x}$ as a vector of returns in which case the following holds

Proposition 2.1. *C is diagonal if and only if $\mathbf{x}$ is uncorrelated.*

Proof. It is well known, see Chu (1973) Theorem 2, that

$$\operatorname{Cov}(x_\ell, x_s) = \left(\int_0^\infty w(t)\, t^{-1} dt\right) C_{\ell s}$$

so that $C_{\ell s} = 0$ *implies that* $\operatorname{Cov}(x_\ell, x_s) = 0$. *Also*

$$\int_0^\infty w(t)\, t^{-1} dt > 0$$

since $\operatorname{Cov}(x_\ell, x_\ell) > 0$ *and* $C_{\ell\ell} > 0$ $\forall \ell$ *by assumption.*$\square$

Thus uncorrelatedness can be explained by the diagonality of C. We turn now to conditional volatility.

Let us consider the variance of x_t conditional on $x_1, x_2, ..., x_{t-1}$. Since the distribution of $(x_1, x_2, ..., x_t)$ is elliptical, we can use Theorem 5 of Chu which tells us that

$$\sigma_t^2 (s_{t-1}) = \int_{s_{t-1}}^{\infty} \frac{f_1 (v)}{f_1 (s_{t-1})} dv \left(C_{t,t} - (C_{t,1}, ..., C_{t,t-1}) C_{1:t-1,1:t-1}^{-1} (C_{1,t}, ..., C_{t-1,t})' \right) \tag{2.4}$$

where $f_1 (s_{t-1})$ is a positive definite function of $(x_1, x_2, ..., x_{t-1})$ defined as in equation (1), $C_{1:t-1,1:t-1}^{-1}$ is a $(t-1) \times (t-1)$ matrix and

$$s_{t-1} = \frac{1}{2} (x_1, x_2, ..., x_{t-1}) C_{1:t-1,1:t-1}^{-1} (x_1, x_2, ..., x_{t-1})' \tag{2.5}$$

In the diagonal case discussed in Proposition 1 we see obvious simplifications arising. Indeed, in this case it follows that

$$s_{t-1} = \frac{x_{t-1}^2}{2C_{t-1,t-1}} + s_{t-2} \tag{2.6}$$

where $t = 2, ..., T$. Equation (2.4) has another interpretation of some interest. It is proportional to the reciprocal of a hazard function with value s_{t-1}; thus in a hazard analysis one should be able to specify a functional form for σ_t^2 as a function of s_{t-1} and then see what the density for $\mathbf{x}$ should be. The function $f_1 (s_{t-1})$ is not the density function of s_{t-1} so that our interpretation of Equation (2.4) in terms of hazard functions does not have the usual probabilistic meaning. With this goal in mind we note the following result.

Proposition 2.2. *Within the elliptical class, for given C, there is a 1-1 relationship between σ_t^2 and the function $f_1 (s_{t-1})$.*

Proof. Chu, Theorem 5, proves the result in one direction. For the reverse direction, without loss of generality, let $C = I$. From equation (2.4)

$$\sigma_t^2 = \frac{\int_{s_{t-1}}^{\infty} f_1 (v) \, dv}{f_1 (s_{t-1})}$$

then for

$$\overline{F}_1(w) = \int_w^\infty f_1(v)\,dv$$

it follows that

$$\frac{1}{\sigma_t^2} = \frac{f_1(s_{t-1})}{\overline{F}_1(s_{t-1})}$$

and

$$\overline{F}_1(s_{t-1}) = \exp\left(-\int_0^{s_{t-1}} \frac{d\ell}{\sigma_t^2(\ell)}\right)\overline{F}_1(0)$$

Hence

$$f_1(s_{t-1}) = \frac{\overline{F}_1(0)}{\sigma_t^2}\exp\left(-\int_0^{s_{t-1}} \frac{d\ell}{\sigma_t^2(\ell)}\right) \tag{2.7}$$

□

Proposition 2 tells us that if we specify σ_t^2 as a function of s_{t-1} then we can deduce $p(\mathbf{x}_{t-1})$ where $\mathbf{x}_{t-1} = (x_1, ..., x_{t-1})'$. In some cases, knowledge of $f_1(s_{t-1})$ will enable us to deduce $f(s)$ and hence we can find $p(\mathbf{x}_t)$ rather than $p(\mathbf{x}_{t-1})$. Prior to presenting the next theorem, we shall consider an example. Consider a simple relationship given by

$$\sigma_t^2 = \frac{v + 2s_{t-1}}{v + t - 3}$$

where $t = 2, 3, ..., T$, then

$$\begin{aligned} f_1(s_{t-1}) &= \frac{\overline{F}_1(0)(v+t-3)}{v\left(1+\frac{2s_{t-1}}{v}\right)}\exp\left(-(v+t-3)\int_0^{s_{t-1}}\frac{d\ell}{v+2\ell}\right) \\ &= \frac{\overline{F}_1(0)(v+t-3)}{v\left(1+\frac{2s_{t-1}}{v}\right)^{\frac{v+t-1}{2}}} \end{aligned} \tag{2.8}$$

and thus

$$pdf(\mathbf{x}_{t-1}) = \frac{K(v+t-3)}{v\left(1+\frac{\mathbf{x}_{t-1}'C_{1:t-1,1:t-1}^{-1}\mathbf{x}_{t-1}}{v}\right)^{\frac{v+t-1}{2}}}$$

To calculate K, the constant of integration, we use $\int pdf(\mathbf{x}_{t-1})\,d\mathbf{x}_{t-1} = 1$.

If we carry out the transformation $\widetilde{z} = C_{1:t-1,1:t-1}^{-\frac{1}{2}} \mathbf{x}_{t-1}$ with $Jacobian = \det |C_{1:t-1,1:t-1}|^{\frac{1}{2}}$ and transform $\widetilde{z}$ to polar coordinates, where we use the fact that $\widetilde{z}$ is now $E_{t-1}(0, I_{t-1})$, and

$$\begin{aligned} z_1 &= r\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{t-2}) \\ z_2 &= r\sin(\theta_1)\ldots\sin(\theta_{t-3})\cos(\theta_{t-2}) \\ &\vdots \\ z_{t-2} &= r\sin(\theta_1)\cos(\theta_2) \\ z_{t-1} &= r\cos(\theta_1) \end{aligned} \tag{2.9}$$

where $0 < r < \infty, 0 < \theta_i \leq \pi$, for $i = 1, ..., t-3$ and $0 < \theta_{t-2} \leq 2\pi$. We apply Theorem 1.5.5 of Muirhead (1982); after integrating out θ_i we arrive at

$$\mathrm{pdf}(r) = \frac{2\pi^{\frac{t-1}{2}}(v+t-3)\,K\,r^{t-2}\det|C_{1:t-1,1:t-1}|^{\frac{1}{2}}}{v\,\Gamma\left(\frac{t-1}{2}\right)\left(1+\frac{r^2}{2}\right)^{\frac{v+t-1}{2}}} \tag{2.10}$$

for $0 < r < \infty$. Changing variables, let $w = \frac{r^2}{v}$

$$\mathrm{pdf}(w) = \frac{2\pi^{\frac{t-1}{2}}\ (v+t-3)\ K\ \det|C_{1:t-1,1:t-1}|^{\frac{1}{2}}\ v^{\frac{t-3}{2}}}{2\,\Gamma\left(\frac{t-1}{2}\right)} \times \frac{w^{\frac{t-3}{2}}dw}{(1+w)^{\frac{v+t-1}{2}}} \tag{2.11}$$

for $0 < w < \infty$. Integrating (2.11) from 0 to ∞, we see that

$$K = \frac{\Gamma\left(\frac{t-1}{2}\right)}{(v+t-3)\,\pi^{\frac{t-1}{2}}\det|C_{1:t-1,1:t-1}|^{\frac{1}{2}}\ v^{\frac{t-3}{2}}\int_0^\infty \frac{w^{\frac{t-3}{2}}dw}{(1+w)^{\frac{v+t-1}{2}}}} \tag{2.12}$$

where the integral on the right side is equal to $B\left(\frac{t-1}{2}, \frac{v}{2}\right)$, see Abramowitz and Stegun (equation 6.2.1, p. 258). Substituting back into equation (2.8),

$$\mathrm{pdf}(\mathbf{x}_{t-1}) = \frac{\Gamma\left(\frac{v+t-1}{2}\right)}{(\pi v)^{\frac{t-1}{2}}\,\Gamma\left(\frac{v}{2}\right)\left(1+\frac{\mathbf{x}_{t-1}' C_{1:t-1,1:t-1}^{-1}\mathbf{x}_{t-1}}{v}\right)^{\frac{v+t-1}{2}} \times \det|C_{1:t-1,1:t-1}|^{\frac{1}{2}}} \tag{2.13}$$

Equation (2.13) describes a multivariate t of dimension $t-1$, characteristic matrix $C_{1:t-1,1:t-1}$ and v degrees of freedom. It can be thought of as a set of normals with covariance matrix $C_{1:t-1,1:t-1}$ all divided by the same independent chi-squared with v degrees of freedom. Applying the argument after Proposition 2, given the full C matrix, allows us to find $f(s)$ and hence we can deduce that $\mathbf{x}_t = (x_1, x_2, ..., x_t)$ has a multivariate t distribution with v degrees of freedom and characteristic matrix C.

3. The GARCH(p,q) Class of Processes

Conditional volatility processes are now widespread in the economic literature. It is then clearly of some interest to examine linear patterns such as

$$\sigma_t^2 = a_t + b_t \; s_{t-1} \tag{3.1}$$

where a_t and b_t may both be functions of t. We present the results for (3.1) in Proposition 3 and then examine the implications of our results for the GARCH(p,q) class of volatility processes.

Proposition 3.1. *If conditional volatility follows a pattern given by equation (3.1) and we know that* $\mathbf{x}_t = (x_1, x_2, ..., x_t)$ *is* $E_t(0, C)$*, then the marginal pdf of* $\mathbf{x}_{t-1}$ *is given by*

$$p(\mathbf{x}_{t-1}) = \frac{a^{1+\frac{1}{b}}\Gamma\left(1+\frac{1}{b}\right)}{\left(\frac{\pi a}{b}\right)^{\frac{t-1}{2}} \det\left|C_{1:t-1,1:t-1}\right|^{\frac{1}{2}} \Gamma\left(\frac{3}{2}+\frac{1}{b}-\frac{t}{2}\right)\left(a+\frac{b}{2}\mathbf{x}_{t-1}' C_{1:t-1,1:t-1}^{-1}\mathbf{x}_{t-1}\right)^{1+\frac{1}{b}}}$$

Proof. *Let* $\mathbf{x}_{t-1} = (x_1, x_2, ..., x_{t-1})$, $\mathbf{x}_{t-1}$ *is* $E_{t-1}(0, C_{1:t-1,1:t-1})$

$$\begin{aligned} f_1(s_{t-1}) &= \frac{\overline{F}_1(0)}{a+b\; s_{t-1}} \exp\left(-\int_0^{s_{t-1}} \frac{dv}{a+b\;v}\right) \\ &= \frac{\overline{F}_1(0)\; a^{\frac{1}{b}}}{(a+b\;s_{t-1})^{1+\frac{1}{b}}} \end{aligned}$$

changing to polar coordinates as in (2.8) and integrating out the θ's leaves us an expression for $\text{pdf}\,(r)$ *where* $r = \sqrt{\mathbf{x}'_{t-1} C^{-1}_{1:t-1,1:t-1} \mathbf{x}_{t-1}}$

$$\text{pdf}\,(r) = \frac{2\pi^{\frac{t-1}{2}} \overline{F}_1\,(0)\;\; r^{t-2}\;\; \det \left| C_{1:t-1,1:t-1} \right|^{\frac{1}{2}}}{\Gamma\left(\frac{t-1}{2}\right) \left(a + \frac{b}{2} r^2\right)^{1+\frac{1}{b}}}$$

$$\therefore \quad \text{pdf}\,(r) = \frac{K\; r^{t-2}}{\left(1 + \frac{br^2}{a}\right)^{1+\frac{1}{b}}}$$

where

$$K = \frac{2\pi^{\frac{t-1}{2}} \det \left| C_{1:t-1,1:t-1} \right|^{\frac{1}{2}} \overline{F}_1\,(0)}{\Gamma\left(\frac{t-1}{2}\right) a^{1+\frac{1}{b}}}$$

Transforming $v = \frac{br^2}{a}$ *we arrive at a Beta distribution,*

$$\text{pdf}\,(v) = \frac{C \left(\frac{a}{b}\right)^{\frac{t-1}{2}} v^{\frac{t-3}{2}} dv}{2\,(1+v)^{1+\frac{1}{b}}}$$

Integrating we see that

$$\overline{F}_1\,(0) = \frac{a\, \Gamma\left(1 + \frac{1}{b}\right)}{\left(\frac{\pi a}{b}\right)^{\frac{t-1}{2}} \det \left| C_{1:t-1,1:t-1} \right|^{\frac{1}{2}} \Gamma\left(\frac{3}{2} + \frac{1}{b} - \frac{t}{2}\right)}$$

and the result follows.□

As a check, if $a_t = \frac{v}{v+t-3}$, $b_t = \frac{2}{v+t-3}$, then Proposition 3 reduces to example (1). In general, Proposition 3 tells us that seemingly simple patterns of conditional volatility, coupled with the assumption of ellipticity or sphericity ($C = I$) lead to rather restrictive results.

Turning now to the GARCH(p,q) class of processes, the great drawback of this model, as mentioned in the introduction, is that so little is known about the joint distribution of its conditioning set. Consider

$$\boldsymbol{\sigma}_t^2 = \boldsymbol{\omega} + \Gamma \boldsymbol{\sigma}_{t-1}^2 + \mathbf{v}_{t-1} \tag{3.2}$$

where $\boldsymbol{\sigma}_t^2 = \left(\sigma_t^2, \sigma_{t-1}^2, ..., \sigma_{t-q+1}^2\right)'$, $\boldsymbol{\omega} = (\omega, 0, ..., 0)$, $\mathbf{v}_{t-1} = \left(\boldsymbol{\beta}'\mathbf{x}_{t-1}^2, 0, ..., 0\right)'$ and $\boldsymbol{\beta} = \left(\beta_1, \beta_2, ..., \beta_p\right)'$, $\mathbf{x}_{t-1}^2 = \left(x_{t-1}^2, x_{t-2}^2, ..., x_{t-p}^2\right)'$

$$\Gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{q-1} & \gamma_q \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Equation (3.2) can be seen as a representation of a GARCH(PL) process. Assuming $\boldsymbol{\sigma}_0^2$ is constant, upon recursive substitution we obtain

$$\boldsymbol{\sigma}_t^2 = \left(I - \Gamma^t\right)\left(I - \Gamma\right)^{-1}\boldsymbol{\omega} + \Gamma^t \boldsymbol{\sigma}_0^2 + \sum_{j=1}^{t} \Gamma^{j-1}\mathbf{v}_{t-j}$$

from which we define

$$a = \mathbf{1}'\left(I - \Gamma^t\right)\left(I - \Gamma\right)^{-1}\boldsymbol{\omega} + \mathbf{1}'\Gamma^t \boldsymbol{\sigma}_0^2$$

where $\mathbf{1}$ is a vector with unit in the first element and zero elsewhere. Also,

$$\begin{aligned} \mathbf{1}'\sum_{j=1}^{t}\Gamma^{j-1}\mathbf{v}_{t-j} &= \sum_{j=1}^{t}\Gamma_{11}^{j-1}\boldsymbol{\beta}'\mathbf{x}_{t-j}^2 \\ &= \sum_{j=1}^{p}\beta_j \tilde{\mathbf{x}}_{t-j}' C^{-1}\tilde{\mathbf{x}}_{t-j} \end{aligned}$$

where $\tilde{\mathbf{x}}_{t-j} = \left(x_{t-j}, x_{t-j-1}, ..., x_{t-j-(t-1)}\right)'$ is a $t \times 1$ vector, $\Gamma^t = \Gamma \times \Gamma \times ...$, t times, Γ_{11} is the $(1,1)$ element of Γ and C^{-1} is a diagonal matrix where C_{jj}^{-1} is given by Γ_{11}^{j-1}. The last expression can be used to derive coefficients $\bar{C}_{jj}^{-1}$ such that

$$\sum_{j=1}^{p}\beta_j \tilde{\mathbf{x}}_{t-j}' C^{-1}\tilde{\mathbf{x}}_{t-j} = \bar{\mathbf{x}}_{t-1}'\bar{C}^{-1}\bar{\mathbf{x}}_{t-1}$$

where $\bar{\mathbf{x}}_{t-1} = \left(x_{t-1}, x_{t-2}, ..., x_{-(p-1)}\right)'$ is a $t+p-1 \times 1$ vector and $\bar{C}^{-1}$ is a diagonal matrix with

$$\bar{C}^{-1}_{jj} = \sum_{i=0}^{j-1} C^{-1}_{j-i,j-i} \text{ for } j = 1, ..., p-1 \tag{3.3}$$

$$\bar{C}^{-1}_{jj} = \sum_{i=0}^{p-1} C^{-1}_{j-i,j-i} \text{ for } j = p, ..., t \tag{3.4}$$

$$\bar{C}^{-1}_{jj} = \sum_{i=0}^{t+p-1-j} C^{-1}_{t-i,t-i} \text{ for } j = t+1, ..., t+p-1 \tag{3.5}$$

Thus

$$\sigma_t^2 = a + \bar{\mathbf{x}}'_{t-1}\bar{C}^{-1}\bar{\mathbf{x}}_{t-1} \tag{3.6}$$

We now state our main result for the general GARCH(p,q) process in the following proposition.

Proposition 3.2. *If σ_t^2 is generated by a GARCH(p,q) process given by equation (3.2) and the matrix $\bar{C}$ is given by equations (3.3), (3.4) and (3.5), then for $\bar{\mathbf{x}}_{t-1} = \left(x_{t-1}, x_{t-2}, ..., x_{-(p-1)}\right)'$, pdf($\bar{\mathbf{x}}_{t-1}$) is given by*

$$pdf\left(\bar{\mathbf{x}}_{t-1}\right) = \frac{a^2\Gamma\left(2\right)}{\left(\pi a\right)^{\frac{t-1}{2}} \det\left|\bar{C}\right|^{\frac{1}{2}} \Gamma\left(\frac{3}{2}+1-\frac{t}{2}\right) \times \left(a+\frac{1}{2}\bar{\mathbf{x}}'_{t-1}\bar{C}^{-1}\bar{\mathbf{x}}_{t-1}\right)^2}$$

where

$$a = \mathbf{1}'\left(I-\Gamma^t\right)\left(I-\Gamma\right)^{-1}\boldsymbol{\omega} + \mathbf{1}'\Gamma^t\boldsymbol{\sigma}_0^2$$

and $\bar{C}^{-1}$ is given by equations (3.3) to (3.5).

Proof. From (3.6) substitute the values of $a = \mathbf{1}'\left(I-\Gamma^t\right)\left(I-\Gamma\right)^{-1}\boldsymbol{\omega} + \mathbf{1}'\Gamma^t\boldsymbol{\sigma}_0^2$ and $b = 1$ in Proposition (3). Noting that this holds for every t, gives us the joint pdf of $\bar{\mathbf{x}}_{t-1}$.$\square$

The special case of GARCH(1,1) is very common in applied work. For lengthy discussions on its distributional properties see Knight and Satchell (1994). We present our results in the form of a corollary.

Corollary 3.3. *If $p = q = 1$ and σ_t^2 is generated by a GARCH(p,q) process given by equation (3.2) and the matrix $\bar{C}$ is given by equation (3.3), then for $\bar{\mathbf{x}}_{t-1} = (x_{t-1}, x_{t-2}, ..., x_0)'$, pdf($\bar{\mathbf{x}}_{t-1}$) is given by*

$$pdf\left(\bar{\mathbf{x}}_{t-1}\right) = \frac{a^{1+\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)}{\left(\frac{a}{b}\right)^{\frac{t-1}{2}}\pi^{\frac{t-1}{2}}\left(\left(\frac{1}{\gamma}\right)^{-\frac{1}{2}t(t-1)}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}+\frac{1}{\beta}-\frac{t}{2}\right)\left(a+\frac{\beta}{2}\mathbf{x}_{t-1}'C^{-1}\mathbf{x}_{t-1}\right)^{1+\frac{1}{\beta}}}$$

where $\bar{C}^{-1}$ is given by equations (3.3) to (3.5) and $a = \frac{(1-\gamma^t)}{1-\gamma}\omega + \gamma^t\sigma_0^2$.

Proof. For $p = q = 1$ from (3.6) we have that

$$\begin{aligned} a &= \mathbf{1}'\left(I-\Gamma^t\right)\left(I-\Gamma\right)^{-1}\boldsymbol{\omega} + \mathbf{1}'\Gamma^t\boldsymbol{\sigma}_0^2 \\ &= \frac{(1-\gamma^t)}{1-\gamma}\omega + \gamma^t\sigma_0^2 \end{aligned}$$

and $\bar{\mathbf{x}}_{t-1}'\bar{C}^{-1}\bar{\mathbf{x}}_{t-1}$ reduces to

$$\beta\ \mathbf{x}_{t-1}'C^{-1}\mathbf{x}_{t-1}$$

where $\mathbf{x}_{t-1} = (x_{t-1}, x_{t-2}, ..., x_0)'$ and

$$\begin{aligned} C_{jj}^{-1} &= \gamma^{t-j} \text{ for } j = t, t-1, ..., 1 \\ C_{ij}^{-1} &= 0 \text{ for } i \neq j \end{aligned}$$

Also

$$\begin{aligned} \det|C| &= \prod_{j=1}^{t}\left(\frac{1}{\gamma}\right)^{t-j} \\ &= \left(\frac{1}{\gamma}\right)^{-\frac{1}{2}t(t-1)} \end{aligned}$$

Substituting in Proposition 3 we obtain the result.$\square$

4. Remarks on Proposition 4

Let us examine Proposition 4 for a GARCH(1,1) as expressed through Corollary 5. The latter tells us that if we have a stochastic process that is elliptical and whose unconditional variance decreases at $\left(\frac{1}{\gamma}\right)^t$ as t increases and whose conditional variance follows a GARCH(1,1), then the pdf of $(x_1, x_2, ..., x_{t-1})$ is a general t distribution. Decreasing Unconditional Volatility is a plausible property of many financial data sets and is a feature of recent equity markets. Typical sets that start near 1987 may appear consistent with this phenomenon, theoretical models of market micro-structure can possess this property as well, see Kyle (1985) or Easley and O'Hara (1992) for example. Empirical estimates of γ are usually less than one, typical values for UK daily returns are near 0.9, β is often near 0.1.

This result is not in contradiction of the work of Nelson (1990) or Bouigeral and Picard (1992) who prove that $\beta+\gamma < 1$ is a necessary and sufficient condition for strong stationarity in a GARCH(1,1) model. We have not restricted β and γ, indeed $\beta + \gamma$ might be less than 1 whilst our σ_t^2 is certainly not stationary. This seeming contradiction is resolved if we note that the above authors assume that $x_t = \sigma_t \varepsilon_t$ where $\varepsilon_t \sim iid\,(0,1)$. In our model ε_t are the marginals which are uncorrelated $(0,1)$ but not independent (except for the normal). However Proposition 4 tells us that the distribution of $x's$ is not normal. We do not investigate the implied properties of the GARCH residuals in Proposition 4.

We conclude here that the assumptions of GARCH modelling of independent ε_t really eliminates the elliptical class as a plausible parent distribution. This is not surprising, it is well known that specifying independent marginal ellipticals will lead to a joint distribution that is non-elliptical, except for the normal case (for a proof see Muirhead, 1982). However, there is a weaker definition of a

GARCH process, see Drost and Nijman (1993) for their definitions of semi-strong and weak GARCH processes. The above model is consisent with such definitions, although Drost and Nijman only consider stationary cases in their paper. Our model, starting with a fixed value, is not stationary, but for certain values, will converge to a stationary process.

5. Conclusion

The conclusion of Propositions 1 to 4 is to demonstrate that certain sets of data that are uncorrelated (Proposition 1), that have a conditional volatility linear in s_t^2 (Proposition 2), decreasing unconditional volatility (Proposition 4) will also be distributed as multivariate t if we assume joint ellipticity. Such a result helps us to understand many contemporary empirical results. Some tracts of financial data exhibit the above characteristics. Researchers often find that residuals of GARCH models are better explained by the t distribution than by the normal. Decreasing volatility is often a pattern of new financial markets. Finally a direct assault on the exact distribution of a process that exhibits GARCH volatility has been thought of as completely intractable, the procedure we develop here of "inverting" the relationship between the conditional volatility and the density leads to a partial solution for a wide class of GARCH models, as demonstrated in Proposition 3. Finally, planned further research could consider the application of our methods to skew-elliptical distributions, see Genton, et al. (2004, 2005).

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