A Simple Multi-Factor "Factor Adjustment" for the Treatment of Credit Capital Diversification

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Abstract

We present a simple adjustment to the single-factor credit capital model, which recognizes the diversification from a multi-factor model. We introduce the concept of a *diversification factor* at the portfolio level, and show that it can be expressed as a function of two parameters that broadly capture the size (sector) concentration and the average cross-sector correlation. The model further supports an intuitive capital allocation methodology through the definition of *marginal diversification factors* at the sector or obligor level. We estimate the diversification factor for a family of models, and show that it can be express in parametric form or tabulated for potential regulatory applications and risk management. As a risk management tool, it can be used to understand concentration risk, capital allocation and sensitivities, stress testing, as well as to compute "real-time" marginal risk.

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1. Introduction

Minimum credit capital requirements under the new Basel II Capital Accord (Basel Committee of Banking Supervision, 2003) are based on the estimation of the 99.9% systemic credit risk for a portfolio (the risk of an asymptotically fine-grained portfolio) under a one-factor Merton type credit model. This model results in a closed form solution, which provides additive risk contributions and is easy to implement. The two key limitations of the model are that it measures only systemic credit risk, and it might not recognize the full impact of diversification.

The first shortcoming has been addressed in an analytical manner, most notably with the introduction of a granularity adjustment (Gordy 2003, 204, Wilde 2001, Martin and Wilde 2002). The second problem is perhaps more difficult to address analytically but has greater impact, especially for institutions with broad geographical and sector diversification. Diversification is one of the key tools for managing credit risk, and it is vital that the credit portfolio framework used to calculate and allocate credit capital effectively models portfolio diversification effects.

Portfolio granularity and diversification within a multi-factor setting can be effectively addressed within a simulation-based credit portfolio framework. However, there are benefits for seeking analytical, closed-form, models both for regulatory applications as well as for credit portfolio management. While the use of simulation-based credit portfolio models is now widespread, they are computationally intensive and may not provide further insights into sources of risk. They are also not efficient for the calculation of sensitivities, stress testing or real-time decision support. Furthermore, the accurate calculation of marginal capital contributions in a simulation framework has proven to be a difficult computational problem, which is currently receiving substantial attention from both academics and practitioners (see Kalkbrener et al. 2004, Merino and Nyfeler, 2004, Glasserman 2005). Analytical or semi-analytical methods generally provide tractable solutions for capital contributions (c.f. Martin et al. 2001, Kurth and Tasche 2003).

In terms of multi-factor credit portfolio modeling, Pykhtin (2004) recently obtains an elegant, analytical multi-factor adjustment, which extends the granularity adjustment technique of Gordy, Martin and Wilde. This method can also be used to compute capital contributions numerically (given its closed form solution to compute portfolio capital). However, the closed-form expressions for capital contributions can be quite intricate.

In this paper, we present an adjustment to the single-factor credit capital model, which recognizes the diversification from a multi-factor setting and which can be expressed parametrically or tabulated for risk management decision support and potential regulatory application. The objective is to obtain a simple and intuitive approximation, based only on a small number of parameters, and which is perhaps less general and requires some numerical estimation work.

To develop the model, we introduce the concept of a *diversification factor*, *DF*, defined as

$$DF = \frac{EC^{mf}}{EC^{Sf}} \quad , \quad DF \leq 1 \tag{1}$$

where EC^{mf} denotes the diversified economic capital from a multi-factor credit model and EC^{sf} is the economic capital arising from the single-factor model.

For a given α percentile level (e.g. $\alpha = 0.1\%$), we seek an approximation to the multi-factor economic capital of the form

$$EC^{mf}(\alpha;\cdot) \approx DF(\alpha;\cdot)\, EC^{sf}(\alpha) \tag{2}$$

with $DF(\alpha;\cdot) \leq 1$ a scalar function of a small number of parameters. Expression (2) allows us to express the diversified capital as the product of the "additive" bottoms-up capital from a one-factor model (e.g. the Basel II model), and a diversification factor (which is a function of say two or three parameters). For potential regulatory use, we may also seek a *conservative* parameterization of the diversification factor.

We estimate the diversification factor for a family of multi-factor models, and show that it can be expressed as a function of two parameters that broadly capture the *size concentration* and the *average cross-sector correlation*.

The diversification factor provides a practical risk management tool to understand concentration risk, capital allocation and correlations. For this purpose, we introduce *marginal diversification factors* at the obligor or sector level, which further account for the diversification contributions to the portfolio.[5] The model (2) supports an intuitive capital allocation methodology, where the

[5] This paper is closely related to Tasche (2006) who further presents a mathematical foundation for the diversification factor and diversification contributions. The author presents a two-dimensional example

diversification contribution of a given sector can be further attributed to three components: the overall portfolio diversification, the sector's relative size, and its cross-sector correlation.

Finally we show how the model can be used in conjunction with a Monte Carlo based multi-factor credit portfolio model (which may already be in use) by implying its parameters. The resulting implied parameters of the *DF* model provide simple risk and sensitivity indicators, which allow us to understand the sources of risk and concentration in the portfolio. The fitted *DF* model can be used further as a risk management tool for capital allocation and sensitivities, as well as for stress testing and real-time computation of marginal capital for new loans or other credit instruments. Since the *DF* model, on its own, is based on the computation of systemic credit capital it only captures sector and geographical concentrations. We further show how it may be augmented with a granularity adjustment to include name concentrations.

The rest of the paper is organized as follows. We first introduce the underlying credit model, the diversification factor and its general analytical justification, and the capital allocation methodology. Thereafter, we explain the numerical estimation of the diversification factor, and provide a parameterization in the context of the Basel II formulae for wholesale exposures. Next, we discuss the application of the model as a risk management tool, its use in conjunction with a Monte Carlo based model and its extension with a granularity adjustment. We illustrate its application with an example. Finally, the paper ends with some concluding remarks

2. A Model for the Diversification Factor

We first introduce the underlying credit model. We then define the concepts of the diversification factor, capital diversification index and average cross-sector correlation. Finally, we discuss capital allocation and risk contributions within the model.

Underlying Credit Model and Stand-Alone Capital

Consider a single-step model with K sectors (sectors can represent an asset class, industry sector, geography, etc.). For each obligor j in a given sector k, the credit losses at the end of the horizon

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which has an analytical solution, and more generally the contribution expressions require integrals of dimension *N-1*, for problems of dimension *N*.

(say, one year) are driven by a single-factor Merton model [6]. Obligor *j* defaults when a continuous random variable Y_j, which describes its creditworthiness, falls bellow a given threshold at the given horizon. If we denote by PD_j the obligor's (unconditional) default probability and assume that the creditworthiness is standard normal, we can express the default threshold by $N^{-1}(PD_j)$. For ease of notation, assume that for obligor *j* has a single loan with loss given default and exposure at default given by LGD_j, EAD_j respectively.

The creditworthiness of obligor *j* is driven by a single systemic factor:

$$Y_j = \sqrt{\rho_k}\ Z_k + \sqrt{1-\rho_k}\ \varepsilon_j \tag{3}$$

where Z_k is a standard Normal variable representing the systemic factor for sector *k*, and the ε_j are independent standard Normal variables representing the idiosyncratic movement of an obligor's creditworthiness. While in the Basel II model all sectors are driven by the same systemic factor *Z*, here each sector can be driven by a different factor. In the general case, the sector factors Z_k are jointly Normal, and denote the sector factor correlation matrix by Q.

To motivate the methodology, it is useful to define a simpler model where the systemic factors are correlated through a single macro-factor, *Z*

$$Z_k = \sqrt{\beta}\ Z + \sqrt{1-\beta}\ \eta_k \ , \qquad k = 1,\ldots,K \tag{4}$$

where η_k are independent standard Normals. We assume first a single correlation parameter β for all the factors, but later relax this assumption and allow for a more general correlation structure.

As shown in Gordy (2003), for asymptotically fine-grained sector portfolios, the stand-alone α-percentile portfolio loss for a given sector *k*, $VaR_k(\alpha)$, is given by the sum of the individual obligor losses in that sector, when an α-percentile move occurs in the systemic sector factor Z_k:

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[6] For consistency with Basel II, we use on a one-period Merton model for default losses. The methodology is general and can be used with other credit models, and can also incorporate losses due to credit migration.

$$VaR_k(\alpha) = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot N\left(\frac{N^{-1}(PD_j) - \sqrt{\rho_k}\ z^{\alpha}}{\sqrt{1-\rho_k}} \right)$$

where z^{α} denotes the α-percentile of a standard normal variable.

Consistent with common risk practices and with the Basel II capital rule, we define the *stand-alone capital* for each sector, $EC_k(\alpha)$, to cover only the *unexpected losses*. Thus, $EC_k(\alpha) = VaR_k(\alpha) - EL_k$, where $EL_k = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot PD_j$ are the expected sector losses.[7] The capital for sector *k* can then be written as

$$EC_k(\alpha) = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot \left[N\left(\frac{N^{-1}(PD_j) - \sqrt{\rho_k}\ z^{\alpha}}{\sqrt{1-\rho_k}} \right) - PD_j \right] \quad (5)$$

Under the Basel II single-factor model, or equivalently assuming perfect correlation between all the sectors, the overall capital is simply the sum of the stand-alone capital for all individual sectors (for simplicity, we omit the parameter α hereafter)

$$EC^{sf} = \sum_{k=1}^{K} EC_k \quad (6)$$

We refer to it as the *single-factor* (*SF*) portfolio capital

The Diversification Factor and Capital Diversification Index

In equation (1) we define the *diversification factor, DF,* as the ratio of the capital computed with the multi-factor model and the SF capital, (equation 6), $DF = EC^{mf} / EC^{sf}$, $DF \le 1$.

As given in equation (2), for a given quantile, we seek to approximate the *DF* by a scalar function of a small number (two or three) of intuitive parameters. We can thus think of the *DF* as "factor adjustment" to the "additive" bottom-up SF capital

$$EC^{mf} \approx DF(\cdot) \times \sum_{k=1}^{K} EC_k$$

[7] The following discussion also holds for VaR (by simply adding back *EL* at the end of the analysis).

Let us first motivate the parameters used for this approximation. We can think of diversification basically being a result of two sources. The first one is the correlation between the sectors. Hence, a natural choice for a parameter in the model is the correlation β of the systemic sector factors Z_k (equation 4) or more generally an "average cross-sector correlation". The second source relates to the distribution of relative sizes of the various sector portfolios. Clearly, one dominating large sector leads to high concentration risk and limited diversification. So we seek a parameter representing essentially an "effective number of sectors". This should account for the size of the sector exposures as well as for their credit characteristics. A large exposure sector with highly rated obligors might not necessarily represent a large contribution from a capital perspective.

Define the *capital diversification index, CDI,* as the sum of squares of the *SF capital weights* in each sector

$$CDI = \frac{\sum_k EC_k{}^2}{\left(EC^{sf}\right)^2} = \sum_k w_k{}^2 \qquad (7)$$

with $w_k = EC_k / EC^{sf}$ the contribution to the SF capital of sector *k*. The *CDI* is the well-known Herfindahl concentration index applied to the SF capital of the sectors (rather than to the exposures, as is more commonly used). Intuitively, it gives an indication of the portfolio capital diversification across sectors (not accounting for the correlation between them). For example, in the two-factor case, the *CDI* ranges between 0.5 (maximum diversification) and one (maximum concentration). The inverse of the *CDI* can be interpreted as an "effective number of sectors" in the portfolio (from a capital perspective). This interpretation of the inverse *CDI* is parallel to the "effective number of loans" interpretation of the inverse Herfindahl (defined on loan exposures) in the original Basel II granularity adjustment proposal (BCBS 2001, ¶436).

It is easy to understand the motivation for the *CDI*. For a set of uncorrelated sectors, the standard deviation of the overall portfolio loss distribution is given by $\sigma_P = \sqrt{CDI} \sum_k \sigma_k$, with σ_P, σ_k the volatilities of credit losses for the portfolio and sector *k*, respectively. If we further assume

that the *credit losses* of each sector are correlated through a single correlation parameter, $\tilde{\beta}$, the volatility of portfolio credit losses is given by [8]

$$\sigma_P = \sqrt{\left(1-\tilde{\beta}\right)CDI + \tilde{\beta}}\sum_k \sigma_k \tag{8}$$

If credit losses were normally distributed, a similar equation to (8) would apply for the credit capital at a given confidence level, $EC^{mf} = DF^N(CDI, \tilde{\beta}) \cdot EC^{1f}$, with $DF^N = \sqrt{\left(1-\tilde{\beta}\right)CDI + \tilde{\beta}}$, the diversification factor for a Normal loss distribution. Figure 1 shows a plot of DF^N as a function of the *CDI* for different levels of the loss correlation, $\tilde{\beta}$. For example, for a *CDI* of 0.2 and a correlation of 25%, the diversified loss volatility from a multi-factor model is about 60% of the SF volatility.

Although credit loss distributions are not Normal, it seems natural to use a similar two-factor parameterization for equation (1):

$$EC^{mf}\left(CDI, \beta\right) \approx DF\left(CDI, \beta\right) \cdot EC^{sf} \tag{9}$$

with the sector systemic factor correlation substituting the loss correlation, given its availability, *a priori*, from the underlying model.

Clearly, we do not expect the parameterization (9) to be exact, nor for the *DF* to follow necessarily the same functional form as DF^N. However, we expect the two parameters to

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[8] One can explicitly obtain the relationship between asset and loss correlations. For the simplest case of large homogeneous portfolios of unit exposures, default probability *PD*, with a single intra-sector asset correlation ρ and correlation of sector systemic factors β, the systemic credit loss correlation is given by

$$\tilde{\beta} = \left[N_2\left(N^{-1}(PD), N^{-1}(PD), \rho\beta\right) - PD^2\right] / \left[N_2\left(N^{-1}(PD), N^{-1}(PD), \rho\right) - PD^2\right]$$

with $N_2(a,b,\rho)$ the standard bivariate normal distribution of random variables *a* and *b* and correlation ρ. Note also that, more generally, the variance of portfolio losses is given by the well-known formula

$$\sigma_p^2 = \sum_{i,j} LGD_i EAD_i LGD_j EAD_j \left[N_2\left(N^{-1}(PD_i), N^{-1}(PD_j), \rho_{ij}\right) - N^{-1}(PD_i)N^{-1}(PD_j)\right]$$

where $\rho_{ij} = \rho_k$ for obligors in the same sector and $\rho_{ij} = \beta\sqrt{\rho_k}\sqrt{\rho_l}$ for obligors in different sectors.

capture broadly the key sources for diversification: homogeneity of sector sizes and cross-sector correlation. Thus, it remains an empirical question whether these two parameters generate a reasonable approximation of the diversification factor. This is further explored in Section 3.

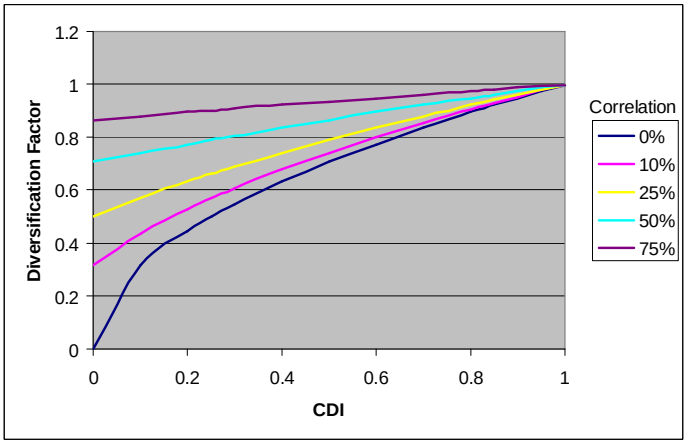

Figure 1. Diversification factor for volatility of losses (Normal distributions)

General Correlation Structure and Average Factor Correlation

So far, we have motivated the model using the simple correlation model implied by equation (4), with all sector systemic factors having the same correlation to an economy-wide systemic factor *Z*. Clearly, this is a very restrictive assumption in practical settings. A less restrictive multi-factor model can also be defined, which assumes a similar structure to equation (4) but where each sector has a different correlation level β_k:

$$Z_k = \sqrt{\beta_k}\ Z + \sqrt{1-\beta_k}\ \eta_k \quad , \qquad k = 1,\ldots,K \tag{10}$$

More generally, we define a factor codependence for the sector factors given by a correlation matrix, *Q*. [9]

A natural choice for a correlation parameter in the general model is some form of average cross-sector correlation (to substitute β in equation 9). From the various possible definitions for an average sector correlation, we choose the following one.

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[9] The model in equation (10) results in only *K* parameters (instead of *K(K-1)/2* for a general correlation matrix), and a correlation matrix *Q* with entries $Q_{ij} = \sqrt{\beta_i \beta_j}$, $j \neq i$.

For a vector of portfolio weights $W = (w_1 ... w_S)^T$, define the *average factor correlation* as

$$\bar{\beta} = \frac{\sum_i \sum_{j \neq i} Q_{ij} w_i w_j}{\sum_i \sum_{j \neq i} w_i w_j} = \frac{\sigma^2 - \delta^2}{\vartheta^2 - \delta^2} \tag{11}$$

where $\sigma^2 = W^T Q W$ is the variance of the random variable given by the weighted sum of the factors, $\delta^2 = \sum_i w_i^2$ and $\vartheta^2 = \left(\sum_i w_i\right)^2$. $\bar{\beta}$ is an average correlation in the sense that

$$W^T B W = W^T Q W = \sigma^2$$

with *B* the correlation matrix with all the non-diagonal entries equal to $\bar{\beta}$.

In a similar way to the *CDI*, we define the average correlation to be also "capital weighted", in order to account for the contributions of each sector (and accounting for both credit quality and size). Thus, for our specific case, we chose the portfolio weights to be the stand-alone capital for each sector, i.e.

$$\delta^2 = \sum_i EC_i^2 \quad \text{and} \quad \vartheta^2 = \left(\sum_i EC_i\right)^2 = \left(EC^{sf}\right)^2 \tag{12}$$

We refer to the general model given by

$$EC^{mf}\left(CDI, \bar{\beta}\right) \approx DF\left(CDI, \bar{\beta}\right) \cdot EC^{sf} \tag{13}$$

with $\bar{\beta}$ defined by equations (11) and (12) as the *DF credit capital model*.

Capital Allocation and Risk Contributions

Under a single-factor credit model, capital allocation is straightforward. The capital attributed to a given sector is the same as its stand-alone capital, EC_k, since the model does not allow further diversification. Under al multi-factor model, the total capital is not necessarily the sum of the stand-alone capitals in each sector. Clearly, the standalone risk of each component does not represent a valid contribution for sub-additive risk measures in general, since it fails to reflect the beneficial effects of diversification. Rather, it is necessary to compute contributions on a marginal basis. The theory behind marginal risk contributions and additive capital allocation is well

developed and the reader is referred elsewhere for its more formal derivation and justification (e.g. Gouriéroux et al 2000, Hallerbach 2003, Kurth and Tasche, 2003, Kalkbrener et al 2004).

After computing the diversification factor in equation (13), one might be tempted simply to allocate back the diversification effect evenly across sectors, so that the total capital contributed by a given sector is $DF \cdot EC_k$. We refer to these as the *unadjusted capital contributions*. These do not account, however, for the fact that each sector contributes differently to the overall portfolio diversification. Instead, we seek an additive capital allocation of the form

$$EC^{mf} = \sum_{k=1}^{K} DF_k \cdot EC_k \tag{14}$$

We refer to the factors DF_k in equation (14) as the *marginal sector diversification factors*.

If *DF* only depends on *CDI* and $\overline{\beta}$, then the economic capital EC^{mf} in equation (13) is a homogeneous function of degree one in the EC_k's (indeed it is homogeneous in the exposure' sizes as well). This is a direct consequence of both the *CDI* and $\overline{\beta}$ being homogenous of degree zero. Applying Euler's theorem, leads to the additive marginal capital decomposition (14) with[10]

$$DF_k = \frac{\partial EC^{mf}}{\partial EC_k} \quad , \quad k = 1,..., K \tag{15}$$

By directly taking the partial derivative on the right side of expression (15), we obtain a closed form expression for the *k*-th marginal diversification factor:

$$DF_k = DF + 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI \right] + 2\frac{\partial DF}{\partial \overline{\beta}} \cdot \frac{1 - \left(EC_k / EC^{sf} \right)}{1 - CDI} \cdot \left[\overline{Q}_k - \overline{\beta} \right] \tag{16}$$

where

[10] Tasche (2006) formally generalizes the diversification factor and the marginal diversification factors introduced here for a general risk measure (he defines the marginal diversification factor of a given position, with respect to a given risk measure, as the ratio of its risk contribution and its stand alone risk).

$$\overline{Q_k} = \frac{\sum_{j \neq k} Q_{kj} EC_j}{\sum_{j \neq k} EC_j}$$

is the average correlation of factor *k* to the rest of the systemic factors in the portfolio. The terms $\partial DF / \partial CDI$ and $\partial DF / \partial \overline{\beta}$ are the slopes of the *DF* surface in the direction of the *CDI* and the average correlation, respectively (they are obtained directly from the parameterized surface as shown later in Section 3). These slopes are non-negative since the portfolios is less diversified as either the *CDI* or the average correlation increase (see for example Figure 1). A brief outline of the derivation of expressions (15) and (16) is given in the appendix. [11]

Expression (16) shows that the marginal capital allocation resulting from the *DF* model leads to an intuitive decomposition of diversification effects (or concentration risk) into three components: *overall portfolio diversification*, *sector size* and *sector correlation*:

$$DF_k = DF + \Delta DF_{Size} + \Delta DF_{Corr} \tag{17}$$

The first term is simply the overall portfolio *DF*. The second terms can be interpreted as an adjustment due to the *"relative size"* of the sector to the overall portfolio. Intuitively, a sector with small stand-alone capital ($EC_k / EC^{sf} < CDI$) contributes, on the margin, less to the overall portfolio capital; thus, it gets a higher diversification benefit, DF_k. The last term is an adjustment due to the sector's correlation to the overall portfolio. Sectors with lower than average correlation to the rest of the systemic sector factors in the portfolio get a higher diversification benefit, as one would expect.

3. Estimating the Diversification Factor Surface

We refer to the function $DF\left(CDI, \beta\right)$ as the diversification factor surface (*DF surface*). We propose to estimate it numerically using Monte Carlo simulations. In general, this exercise

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[11] When one defines the average correlation as an arithmetic average, $\overline{\beta} = \sum (EC_k / EC^{sf}) \cdot \beta_k$, the resulting formula for the marginal sector diversification factor is simpler and given by

$$DF_k = DF + 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI \right] + \frac{\partial DF}{\partial \overline{\beta}} \cdot \left[\beta_k - \overline{\beta} \right]$$

Although simpler, this definition has some undesirable properties which result in inconsistencies.

requires the use of a multi-factor credit portfolio application (which might itself use simulation to obtain a capital estimate). The estimated surface can then be used generally in parametric or for economic capital (EC) calculations in a multi-factor setting, without recourse to further simulation. Note that, for regulatory use, we might seek to estimate a *conservative* diversification factor surface, by finding reasonable upper bounds for the *DF*.

This section presents the general estimation methodology and illustrates its application to a portfolio of wholesale exposures (corporates, banks and sovereign) in the context of the Basel II formula. We first explore the feasibility of the methodology and illustrate it in detail for a two-dimensional case and a fixed cross-sector average correlation. We extend the results to multiple sectors and correlation levels. As the number of sectors is increased and correlations are changed, this exercise demonstrates the basic characteristics of the surface, the approximation errors and the robustness of the results. Thereafter, a parametric form of the overall surface is provided. Finally, we show that the estimated surface produces accurate capital estimates for the more general (and realistic) case where sectors have different correlations to the overall portfolio.

***DF* Surface Estimation Methodology**

The general estimation methodology can be summarized as follows. Assume in each simulation, a set of homogeneous portfolios representing each sector. Each sector is assumed to contain an infinite number of obligors with the same *PD* and *EAD*. Without loss of generality, we set *LGD* = 100%, and the total portfolio exposure equal to one, $\sum EAD_k = 1$, and assume that all loans in the portfolio have a maturity of one year The following numerical experiments are then performed:

- Run a large number of portfolios, varying independently in each run:
  - The number of sectors and sizes of each sector
  - PD_k , EAD_k , ρ_k , $k = 1,...,K$
  - The average factor correlation $\overline{\beta}$
  - For each portfolio
    - Compute the stand-alone capital for each sector, EC_k $(k = 1,...,K)$ the single-factor capital for the portfolio, EC^{sf} , and *CDI* (equations 5, 6 and 7)
    - Compute the "true" EC^{mf} from the multi-factor Monte-Carlo based model[12]

[12] We use a MC-based portfolio model, although a semi-analytical model can be used alternatively.

- Plot the ratio of (EC^{mf} / EC^{sf}) vs. the *CDI* and average sector correlation $\overline{\beta}$

- Estimate the function *DF* (*CDI*, $\overline{\beta}$) by fitting a parametric function through the points

As a simple example, Figure 2 presents the plot for *K*=2 to 5 for a fixed $\overline{\beta}$ = 25% and random independent draws with $PD_k \in [.02\%,20\%]$, $\rho_k \in [2\%,20\%]$. The dots represent portfolios with different parameters. The colours of the points represent the different number of sectors. Simply for reference, for each *K*, we also plot the convex polygons enveloping the points. Figure 2 shows that, in the case where all parameters are varied independently, the approximation is not perfect; otherwise all the points would lie on a line (not necessarily straight). However, all the points lie within a well bounded area, suggesting it as a reasonable approach. A function *DF* can be reliably parameterized either as a fit to the points or, more conservatively, as their envelope. The latter might be more desirable from a regulatory perspective, and can also be estimated with standard statistical methods. For example, for a *CDI* of 0.5, a diversification factor of 80% results in a conservative estimate of the capital reduction incurred by diversification, while the mean fit of the surface would lead to a *DF* of 74%.

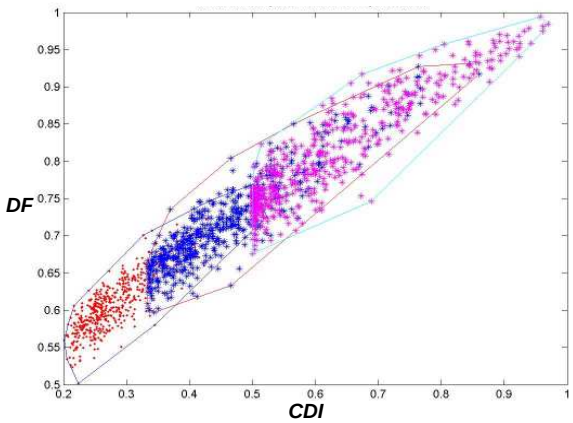

Figure 2. *DF* as a function of the *CDI* (*K*=2 to 5, and $\overline{\beta}$=25%)

This example illustrates the estimation methodology for a general setting where sector *PDs*, exposures and intra-sector correlations are varied independently. Even in this case, two parameters (*CDI*, $\overline{\beta}$) provide a reasonable explanation of the diversification factor. One can get a tighter approximation, by either searching for more explanatory variables, or by constraining the set over which the approximation is valid. In practice, *PDs* and intra-sector correlations do not vary independently and they might only vary over small ranges. For example, under the Basel II

formula, the asset correlation is either constant on a given asset class (e.g. revolving retail exposures, at 4%) or varies as a function of *PDs* (e.g. wholesale exposures).[13] See also Lopez (2004), which provides evidence that the average asset correlation is a decreasing function of *PD* and an increasing function of asset size.

For the rest of this section, we focus on the estimation of the *DF* surface for the case of wholesale exposures (corporates, banks and sovereign) in the context of Basel II.

DF for Wholesale Exposures – Two-Factors and Constant Correlation Level

Consider a portfolio of wholesale exposures in two homogeneous sectors, each driven by a single factor model. Assume a cross-sector correlation $\overline{\beta}$ = *60%*. We perform a simulation of three thousand portfolios. The *PDs* are sampled randomly and independently from a uniform distribution in the range [0,10%]. Asset correlations for each sector are given as a function of *PD*s from the Basel II formula for wholesale exposures without the firm-size adjustment. For each of the 3,000 portfolios, the multi-factor EC is calculated using a MC simulation with one million scenarios on the sector factors, and assuming granular portfolios. EC is estimated at the 99.9% percentile of credit losses (net of the expected losses).

Figure 3 plots the *DF* as a function of the *CDI* for the simulated portfolios. With two factors, the *CDI* ranges between 0.5 (maximum diversification) and 1 (maximum concentration). There is a clear relationship between the diversification factor and the *CDI*, and a linear model fits the data very well, with an R^2 of 0.97. We can express the diversification factor as[14]

$$DF(CDI, \overline{\beta} = 0.6) = 0.6798 + 0.3228 \cdot CDI$$

The figure further presents in tabular form the results of the regression. Accounting for maximum diversification, the capital savings are 16% .

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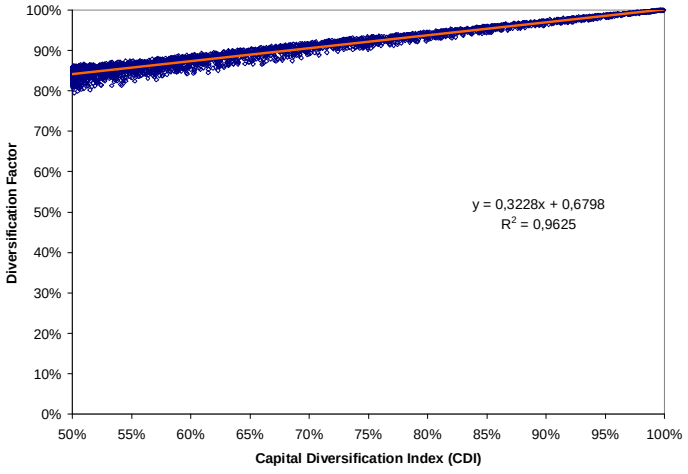
[13] In this case, the asset correlation is given by

$$\rho = 0.12\left(\frac{1-e^{-50PD}}{1-e^{-50}}\right) + 0.24\left(1-\frac{1-e^{-50PD}}{1-e^{-50}}\right)$$

[14] Similarly, one can obtain the parametric envelop of the data, to get a more conservative adjustment.



CDI	Diversification Factor
50%	84%
55%	86%
60%	87%
65%	89%
70%	91%
75%	92%
80%	94%
85%	95%
90%	97%
95%	99%
100%	100%
Intercept	0.6798
slope	0.3228
R^2	0.97

Figure 3. Two-factor diversification factor as a function of *CDI* ($\overline{\beta}$ = 60%)

Figure 4 displays, for all simulated portfolios, the actual EC against that estimated from the *DF* model (using the regression in Figure 3). There is clearly a close fit between the two models, with the standard error of the estimated *DF* model of only 10 basis points.

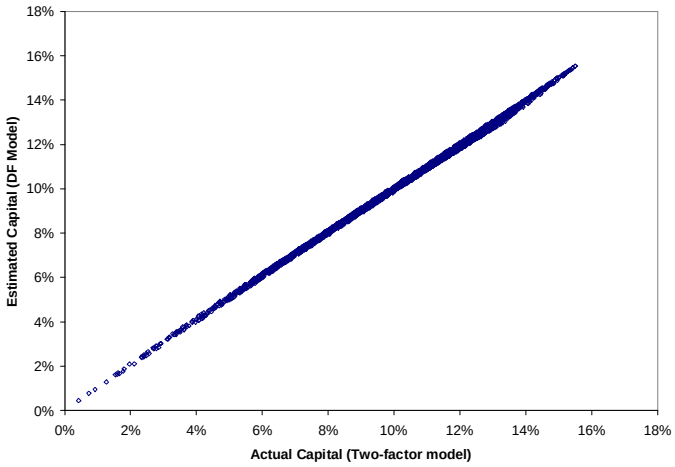


Figure 4. Capital from *DF* model vs. actual two-factor capital ($\overline{\beta}$ =60%)

Estimation of *DF* for Varying Number of Sectors and Average Factor Correlation levels

Next, we investigate the behaviour of the *DF* as a function of the number of factors and average cross-sector correlation levels. First, consider portfolios of wholesale exposures consisting of *k* homogeneous sectors, *K=2*, 3,…,10, and average cross-sector correlation fixed at $\overline{\beta}$ = *60%* .We follow the same estimation procedure using a simulations of three thousand portfolios, for each *K*.

Figure 5 plots the nine regression lines. At this correlation level, a linear model fit the data well in all cases with R^2 ranging between 96-98%, and standard approximation errors of 10-11 bps. Table 1 tabulates the estimated *DF* for each *K*, and gives the coefficients of the regressions, as well the average over the range of factors.

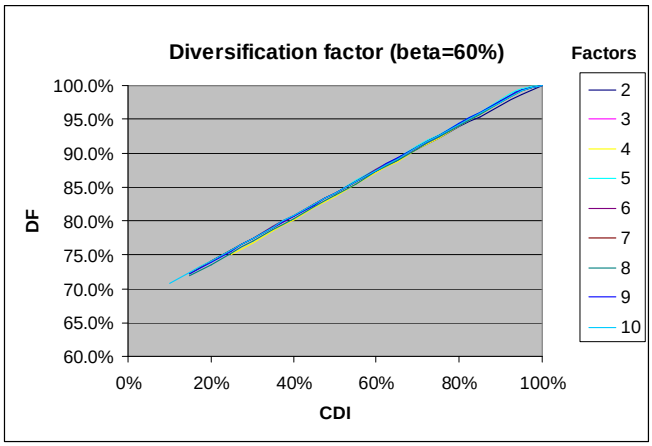


Figure 5. *DF* model regression lines for *K=2, …, 10* ($\overline{\beta}$ =60%).

CDI \ Factors	2	3	4	5	6	7	8	9	10	Average
10%									70.7%	70.7%
15%						72.3%	71.9%	72.2%	72.4%	72.2%
20%				74.0%	74.0%	74.0%	73.6%	73.9%	74.0%	73.9%
25%			75.0%	75.7%	75.7%	75.7%	75.3%	75.6%	75.7%	75.5%
30%			76.8%	77.4%	77.4%	77.4%	77.0%	77.3%	77.4%	77.2%
35%		79.1%	78.5%	79.1%	79.1%	79.0%	78.7%	79.0%	79.1%	78.9%
40%		80.7%	80.2%	80.8%	80.8%	80.7%	80.4%	80.7%	80.8%	80.6%
45%		82.4%	81.9%	82.5%	82.5%	82.4%	82.1%	82.4%	82.4%	82.3%
50%	84.1%	84.1%	83.7%	84.2%	84.2%	84.1%	83.8%	84.1%	84.1%	84.0%
55%	85.7%	85.8%	85.4%	85.9%	85.8%	85.8%	85.5%	85.8%	85.8%	85.7%
60%	87.3%	87.4%	87.1%	87.6%	87.5%	87.5%	87.2%	87.5%	87.5%	87.4%
65%	89.0%	89.1%	88.8%	89.3%	89.2%	89.2%	88.9%	89.2%	89.1%	89.1%
70%	90.6%	90.8%	90.6%	91.0%	90.9%	90.8%	90.6%	90.9%	90.8%	90.8%
75%	92.2%	92.5%	92.3%	92.7%	92.6%	92.5%	92.4%	92.6%	92.5%	92.5%
80%	93.8%	94.1%	94.0%	94.4%	94.3%	94.2%	94.1%	94.3%	94.2%	94.1%
85%	95.4%	95.8%	95.7%	96.1%	95.9%	95.9%	95.8%	96.0%	95.9%	95.8%
90%	97.0%	97.5%	97.5%	97.8%	97.6%	97.6%	97.5%	97.7%	97.5%	97.5%
95%	98.6%	99.1%	99.2%	99.5%	99.3%	99.3%	99.2%	99.4%	99.2%	99.2%
100%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Intercept	0.6798	0.6734	0.6641	0.6722	0.6731	0.6726	0.6675	0.6706	0.6732	0.6718
slope	0.3228	0.3349	0.3449	0.3397	0.3369	0.3368	0.3413	0.3406	0.3359	0.3371
R^2	96.3%	96.9%	97.2%	97.6%	98.0%	97.9%	97.9%	98.0%	98.1%	

Table 1. Tabulated results for the *DF* model, *K=2,…10* ($\overline{\beta}$ =60%).

While a linear function works well at this correlation level, further experiments show that the *DF* is non-linear at lower correlation, and its curvature increases with decreasing correlation. Figure 6

illustrates this point by plotting the *DF* for three levels of correlation. One can get some intuition on the curvature of the *DF* surface by revisiting the functional form for portfolio loss volatility, Equation (8) (note also the similarity between Figures 1 and 6).

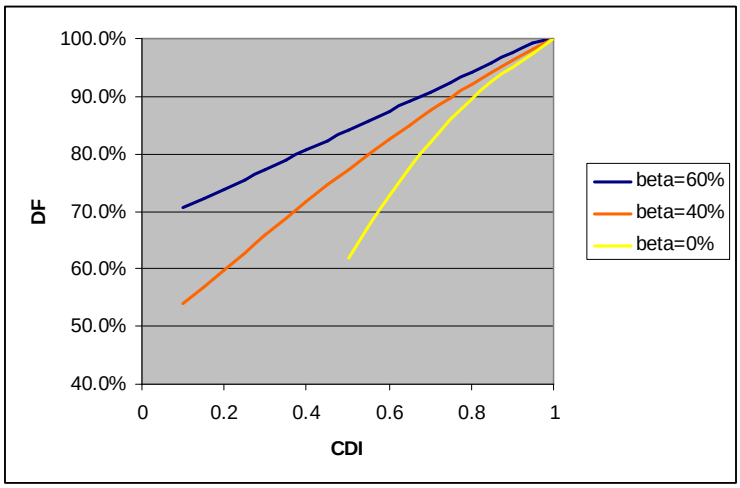


Figure 6. *DF* model for various correlation levels.

Parametric Estimation of *DF* Surface

Based on the insights gained in the previous exercises, we now estimate a parametric *DF* surface as a function of both the *CDI* and average cross-sector correlation. We search for a polynomial function in *CDI* and $\overline{\beta}$ of the form

$$DF = P_n(CDI, \overline{\beta})$$

with

$$P_n(CDI, 1) = 1 \text{ and } P_n(1, \overline{\beta}) = 1 \tag{18}$$

The first restriction states that when the inter-sector correlations are one, the model reduces to the single-factor model and *DF* = 1. The second restriction refers to the case where there is essentially one sector and hence no diversification. The restrictions (18) suggest the following functional form [15]

$$P_n(CDI, \overline{\beta}) = 1 + \sum_{i \geq 1, j \geq 1} a_{i,j} (1 - \beta)^i (1 - CDI)^j \tag{19}$$

More specifically, we investigate the second order approximation (in each variable):

[15] Note that a Taylor series expansion of equation (8) suggests a similar functional form.

$$P_2(CDI,\overline{\beta}) = 1 + a_{1,1}(1-\beta)(1-CDI) + a_{2,1}(1-\beta)^2(1-CDI)^1 + a_{1,2}(1-\beta)^1(1-CDI)^2 + \\ + a_{2,2}(1-\beta)^2(1-CDI)^2 \quad (20)$$

We proceed to estimate the *DF* surface defined by equation (20) as follows. We randomly simulate 22,000 portfolios with up to 10 sectors (*CDI* in [0.1, –1]). For each portfolio, the weights and *PD*s in each sector are sampled randomly, as well as the average inter-sector correlation $\overline{\beta}$ (sampled uniformly between 0 and 1). Figure 7 gives the 3-D plot of the *DF* for all simulated portfolios as a function of the *CDI* and $\overline{\beta}$.

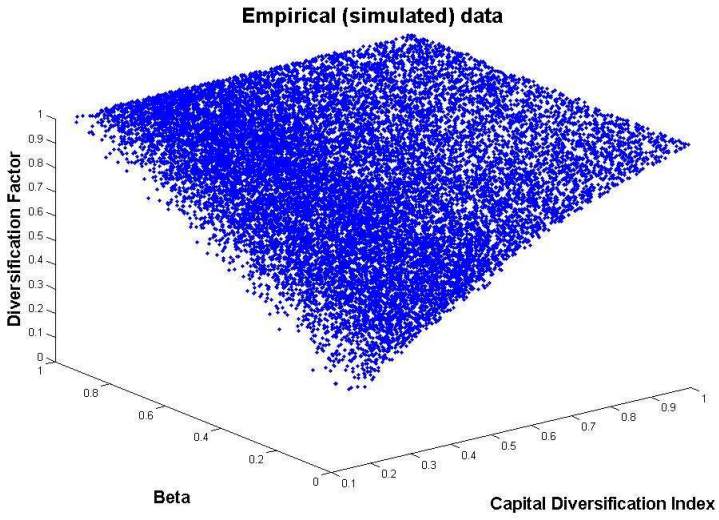


Figure 7. *DF* for simulated portfolios.

Table 2 presents the specification of equation (20) resulting from fitting the simulated data. The approximation is excellent, with R^2 coefficients of 99.4% and above and a volatility of errors of 11 bps. Note that all the coefficients have tight bounds and also are significant, except for the third one, where we cannot reject the null hypothesis that it is zero.

	Coefficients	Standard Error	t Stat	Lower 95%	Upper 95%
A11	-0.845	0.0046	-185	-0.854	-0836
A21	0.417	0.0059	71	0.406	0.429
A12	-0.0118	0.0072	-1.6	-0.0260	0.0023
A22	-0.467	0.0093	-50	-0.485	-0.449

Regression Statistics	
Multiple R	0.997
R Square	0.994
Adjusted R Square	0.994
Standard Error	0.0107
Observations	22,000

Table 2. Estimated parametric *DF* model (equation 20).

Table 3 presents the final specification of the model, restricting the third coefficient to zero.

	Coefficients	Standard Error	t Stat
A11	-0.852	0.0009	-911
A21	0.426	0.0019	225
A12	0		
A22	-0.481	0.0023	-205

Table 3. Final Estimated (simplified) parametric *DF* model (equation 20).

Figure 8 presents the surface in equation (20) with the coefficients in Table 3. Figure 9 compares the estimated surface with the simulated portfolios (left side), and shows the model estimation errors for the simulated portfolios. [16]

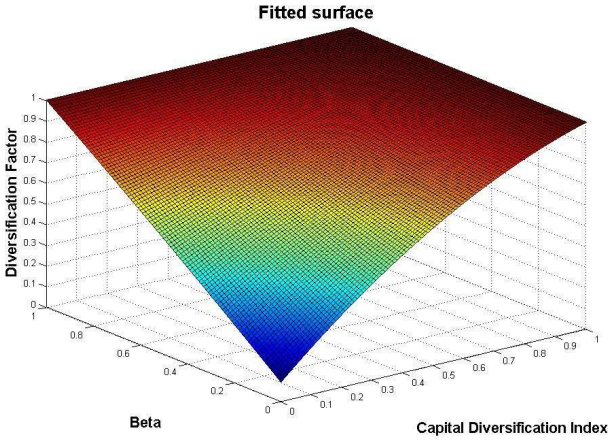


Figure 8. Estimated *DF* Surface.

[16] Additional estimation exercises were performed. For example, a specification where the constant polynomial term is estimated (instead of fixing it at one) yields an estimated constant of *1.0021*. However, the explanatory power increases only marginally. Another model with the additional constraint $DF = P_n(0,0) = 0$ implies a restriction of the form: $-1 = a_{1,1} + a_{2,1} + a_{1,2} + a_{2,2}$. At the expense of fitting the surface to an area not relevant in practice (infinite number of independent sectors), this model results in reduced explanatory power.

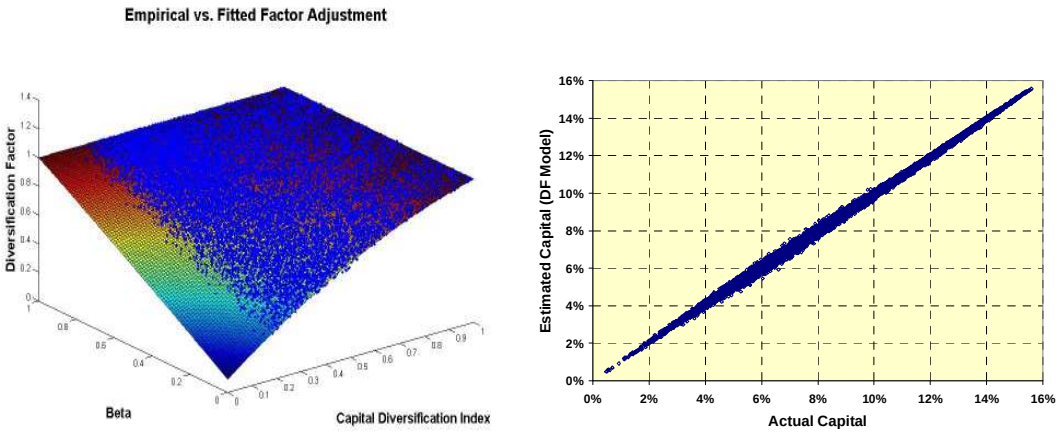


Figure 9. Simulated portfolios and the estimated *DF* Surface.

Estimated *DF* Surface for Non-Homogeneous Factor Correlations

The model developed in this paper suggests parameterizes the *DF* surface in terms of a single average correlation (and the *CDI*). As a practical exercise, the surface estimated above assumes a simplistic correlation structure with a single (average) correlation for all sectors. Since, in practice, different sectors likely present different levels of correlation to the overall portfolio, it is important to assess the impact of this assumption on the estimated model.

In order to test this impact we perform the following (out of sample) exercise. We simulate 22,000 new portfolios with up to 10 sectors (as before, with random weights and *PD*s). For each portfolio we assume a multi-factor model defined by equation (10)

$$Z_k = \sqrt{\beta_k}\ Z + \sqrt{1-\beta_k}\ \eta_k \quad , \qquad k = 1,\ldots,K$$

where each sector has a different correlation level β_k. Thus, in addition, we randomly simulate the β_k's for each sector and calculate the average inter-sector correlation (equation 11).

Figure 10 shows the actual diversification factor for all simulated portfolios (as the ratio of their true EC to their single-factor capital) as well as the estimated *DF* surface above. It also plots the true EC vs. the estimated EC from the model (equation 20 and Table3). Contrasting Figures 9 and 10, the out-of-sample model performance (with non-homogeneous sector correlations) is very good and generally of the same quality as the in-sample performance. The volatility of the errors is only 14bps (with a mean error of 4bps), only slightly higher than the in-sample errors.

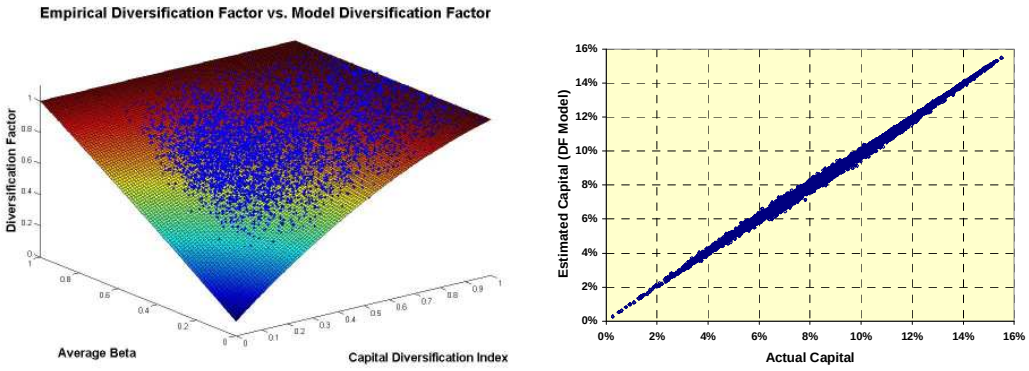


Figure 10. Out-of-sample simulated portfolios (non-homogeneous correlations).

4. The Diversification Factor as a Management Tool

The *DF* model provides a simple alternative to Monte Carlo (MC) simulation for the computation of portfolio EC. Due its analytical tractability as well as its intuitive parameters and capital allocation, the model can be used as a risk management tool for

- Understanding concentration risk and capital allocation
- Identifying capital sensitivities
- Stress testing
- "Real-time" marginal risk contributions for new deals or portfolios

In this section, we explore these applications. First, we summarize the parameters of the model and analytical sensitivities. Next, we review its application for stress testing. Thereafter, we show how the model can be used for risk management in conjunction with an existing MC-based multi-factor credit portfolio model, by computing its implied parameters. While the *DF* model, as presented, covers only systemic risk and hence measures sector and geographical concentrations, we outline how the framework might be extended, in conjunction with the granularity adjustment, to cover also name concentrations. Finally, we present a stylized example to illustrate the calculation of implied parameters, and show the model's application for sensitivities, stress testing and real-time marginal capital calculation of new deals.

Summary of Model Parameters and Sensitivities

The *DF* model's parameters can be seen as intuitive risk and concentration indicators. We divide these parameters into: *sector-specific indicators, portfolio capital indicators, capital contributions and correlation indicators.* For completeness, we summarize them in Table 4.

Sector specific (sectors k=1,…,K) [17]		***Portfolio capital***		***Marginal capital contributions (sectors k=1,…,K)***	
Inputs					
ρ_k	Intra-sector (asset) correlation	EC^{sf}	Capital one-factor (undiversified)	β_k	Sector factor correlation weights
$\overline{PD}_k$	average default probability	CDI	Capital diversification index	$\overline{Q}_k$	Average correlation of a sector factor to the other sectors
$\overline{EAD}_k$ $\overline{LGD}_k$	Average exposure, loss given default	$\overline{\beta}$	Average cross sector correlation		
Outputs					
EC_k	Stand-alone capital	DF	Diversification factor	$DF_k \cdot EC_k$	Marginal capital contribution
		EC^{mf}	Economic capital (diversified)	DF_k	Sector diversification factor $DF_k = DF + \Delta DF_k^{size} + \Delta DF_k^{corr}$
				ΔDF_k^{size}	Sector size diversification component
				ΔDF_k^{corr}	Sector's correlation diversification component

Table 4. Parameters and risk indicators of *DF* model

Due to its analytical tractability, closed form formulae are obtained for the sensitivities of the *DF* or the EC to every input parameter of the model. The sensitivities of the *DF* to the average cross-sector correlation and the *CDI*

$$\frac{\partial DF}{\partial \overline{\beta}}, \quad \frac{\partial DF}{\partial CDI}$$

are given directly as the slopes from the estimated *DF* surface. It is straightforward (through the chain rule) to get the sensitivities of the *DF* or EC to the parameters such as sector single-factor (SF) capital (EC_k), exposures, *PDs*, *LGDs*, or sector correlation parameters (ρ_k $\overline{Q}_k$, β_k). Thus, for example, the change in EC per unit of sector factor correlation for *k*-th sector is given by

[17] Commonly, the (exposure-weighted) average *EAD* and *LGD* for each sector are computed, and the average *PD is* implied from the actual calculation of expected losses.

$$\frac{\partial EC^{mf}}{\partial \beta_k} = \left(\frac{\partial EC^{mf}}{\partial \bar{\beta}}\right) \cdot \left(\frac{\partial \bar{\beta}}{\partial \beta_k}\right) = df^c \cdot \left(\frac{EC^{sf}}{\left(EC^{sf}\right)^2 - \delta^2}\right) \cdot \left(\frac{\sum_{i \neq k} \sqrt{\beta_i}}{\sqrt{\beta_k}}\right), \quad (k = 1,\ldots,K)$$

where $df^c = \partial DF / \partial \bar{\beta}$ is the slope in the *DF* surface in the direction of the average correlation.

In addition to portfolio EC, sensitivities can also be explicitly written for the marginal diversification factors and marginal EC for any given sector or exposure in a sector.

Stress Testing and Real-Time Marginal Capital

Stress testing of EC and real-time marginal capital calculations for new deals are difficult when using a portfolio model which requires MC simulation. Each stress test or marginal capital calculation for a new deal or portfolio requires, in essence, an additional MC simulation. For example, measuring the impact of an increase in *PD* for a given sector requires to re-run the MC simulation for the entire portfolio and comparing the capital with the base case. Adding a new loan or new portfolio also requires simulating the whole portfolio in order to aggregate the losses over each scenario and account for correlations. [18]

The *DF* model offers analytical formulae that can be used to calculate the capital for new portfolios or changes in parameters. It also provides intuitive parameters, sensitivities and a capital allocation breakdown which can be used to explain the results further and manage the portfolio. As an example, consider correlations stress testing. Given that the correlation structure generally involves a large number of parameters (which are also closely related to each other), it is not straightforward to define reasonable, and useful, correlation stress tests. The *DF* model provides clear guidance for correlation stress testing. First, in addition to intra-sector correlations parameters (ρ_k), it defines measures of average portfolio cross-sector correlation ($\bar{\beta}$) and of average cross sector correlation of each sector to the rest of the portfolio ($\bar{Q}_k$, β_k). Thus, we can explicit calculate sensitivities or stress tests of the portfolio EC, or the capital contribution of any given sector (or the diversification factors), to changes in:

- A sector's intra-sector correlation (ρ_k)

[18] It is possible to implement a MC method with efficient short cuts for these new simulations. This is difficult to implement and still orders of magnitude slower than analytical solutions.

- Average cross-sector correlation across the portfolio ($\overline{\beta}$)
- Average correlation of a sector to the rest of the portfolio ($\overline{Q_k}$)
- Sector factor loading for a sector (β_k) (in the simplified model, equation 10)

The impact of correlations on sector capital contributions can also be explained through the decomposition of the marginal diversification factor (equations 16 and 17). Finally, stress tests with discrete changes in correlations are computed instantaneously using the closed-from EC formula. An example at the end of this section illustrates correlation stress testing.

Implied Parameters from a Monte Carlo based Multi-Factor Portfolio Model

Many institutions today have implemented advanced (internally built or vendor) multi-factor credit portfolio models based on MC simulation.[19] The *DF* model provides a simple alternative approximation to a full MC simulation. In this section, we show how the *DF* model can also be used *in conjunction* with an existing, and perhaps more detailed, multi-factor credit portfolio model by calculating its *implied* parameters.[20] Furthermore, the *DF* model as presented so far only covers systemic risk and, hence, captures only sector and geographical concentrations. In this context, the framework can be extended using the granularity adjustment concept to include name concentrations for non-granular portfolios, The fitted *DF* model, with its implied parameters, can be used then as a simpler and much faster model to

- Understand the problem better, provide concentration measures and communicate risks
- Obtain analytical sensitivities and perform systematic portfolio stress testing
- Perform real-time marginal capital calculations

Assume that the institution already has a (calibrated) credit portfolio model which it runs periodically to obtain EC via a MC simulation. Also, for ease of exposition, assume that the

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[19] A large portion of the implementation effort is spent on estimating all the relevant parameters of the portfolio model (*PD*s, *LGD*s, *EAD*s, correlations – intra-sector and inter-sector, etc.). The estimation of these parameters is beyond the scope of this paper.

[20] In this sense, the fitted *DF* model is akin to a Black-Scholes model with an implied volatility surface or a CDO copula model with implied correlations.

portfolio can be divided into *K* homogeneous sectors (not necessarily granular), each with a single *PD, EAD* and *LGD* (in practice this latter assumption is easy to relax).[21] The simpler, multi-factor *DF* model given by equations (3) and (10), requires *2K* correlation parameters (ρ_k, β_k). Thus, we require as many statistics on the systemic EC to be calculated from the MC model to imply a *DF* model which produces (locally) the same results.[22] In addition, we would like to extend the model to capture idiosyncratic risk. Thus, we assume that the credit portfolio MC model provides estimates of

- Expected losses, systemic EC and total EC (including idiosyncratic risk) for the portfolio (i.e. it computes the EC of the portfolio under the assumption of an infinitely granular portfolio and also accounting for idiosyncratic risk).

- Marginal (systemic) EC contributions for each sector

We refer in this section to these outcomes of the MC model as the "true" values.

The implied *DF* model is obtained in two steps. The first step calculates the implied parameters of the *DF* model in equation (13) from the true systemic capital numbers obtained from the MC model. The second step adds an idiosyncratic component to the model in the form of the granularity adjustment. We now describe these two steps.

First Step: Systemic Risk and Implied DF Model

The algorithm to solve the inverse problem for the *DF* model's implied parameters is as follows:

- Get for each sector portfolio $k=1,\ldots,K$, its "true" stand-alone systemic EC (e.g. from the MC simulation of each sector separately).
- Solve for the implied intra-sector correlation, ρ_k, from equation (5). This provides an indication of the average correlation (even for non- homogeneous portfolios).

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[21] Sector homogeneity is not a requirement. Note that equation (4) does not require single *PDs, EADs* and *LGDs* for each sector.

[22] The general *DF* credit capital model allows for defining a full correlation matrix *Q*, instead of the simpler model in equation 10. Obtaining an implied *DF* model, in this case, may require a much larger number of parameters, and this is more readily obtained using an optimization procedure.

- From the *K* stand-alone systemic EC for each sector, EC_k , compute the systemic single-factor capital for the overall portfolio, EC^{sf} , and the *CDI* (equations 6 and 7).
- Get the overall "true" systemic EC for the portfolio, EC^{mf} (from MC simulation).
- Use the estimated *DF* surface to solve for the implied average correlation, $\overline{\beta}$ from equation (13):

$$EC^{mf}\left(CDI,\beta\right)=DF\left(CDI,\overline{\beta}\right)\cdot EC^{sf}$$

- Get the *K* "true" marginal EC contributions for each sector, $DF_k \cdot EC_k$ (from MC simulation)
- Solve for the implied inter-sector correlation parameters $\overline{Q}_k$ and β_k from the marginal capital contributions (equations 15 and 16) [23]

We can see from this algorithm that the *DF* model basically provides a map from the correlation parameters to various systemic capital measures:

- Single-factor systemic EC ←→ intra-sector correlations
- Overall systemic EC (or *DF*) ←→ average cross-sector correlation
- Marginal systemic capital contributions ←→ individual cross-sector correlations

Second Step: Idiosyncratic Risk and Implied Granularity Adjustment

To add idiosyncratic risk (and handle name concentrations), we can use the concept of the *granularity adjustment* (*GA*) introduced by Gordy (2003, 2004). Thus, a natural generalization of the *DF* model for non-granular portfolios is obtained by adding a *GA* to equation (2):

$$EC \approx DF\left(CDI,\overline{\beta}\right)\cdot EC^{sf}+GA^{DF} \qquad (21)$$

In the context of using the model in conjunction with a MC-based model, the *GA* can be implied directly from the "true" total EC and the "true" systemic EC simply as

$$GA^{DF}=EC^{True}-EC^{Sys}$$

[23] An implied correlation matrix *Q* can be more generally obtained from the $\overline{Q}_k$'s using an optimization procedure. While multiple solutions might exist, one can use simple criteria to choose one. Also, the implied correlation structure implied by the β_k 's in equation (10) may only result in an approximate match.

For a homogenous portfolio, Gordy (2003) shows that the *GA* is proportional to *1/n* , where *n* is the number of loans in the portfolio. For non-homogeneous portfolios[24], Gordy proposes a two-step method: first, map the actual portfolio to a homogeneous "comparable portfolio" by matching moments of the loss distribution. Second, determine the *GA* for the comparable portfolio. The same add-on is applied to the capital charge for the actual portfolio.

Using this intuition, we may simply write the *GA* in equation (21) as

$$GA^{DF} = b^{GA} \cdot \left(\frac{1}{n^*} \right) \tag{22}$$

The effective number of loans n^* may be calculated bottom-up, as explained in Gordy (2003). The slope b^{GA} can be then implied from the "true" total EC, the "true" systemic EC and the effective number of loans as:

$$b^{GA} = \left(EC^{True} - EC^{Sys} \right) \cdot n^* \tag{23}$$

In this sense, we can complement the *CDI* (which is essentially a measure of sector capital concentration) with the effective number of loans n^* (or its inverse, Herfindahl-like index) as an overall portfolio *name concentration* measure.

Before concluding this section, some comments on the *GA* are appropriate. The original *GA* method is essentially a second order Taylor series expansion of the quantile (around the "infinitely granular" portfolio for the single-factor model).[25] Pykhtin (2004) developed an extension for multiple factors. In this model, portfolio EC is essentially represented as the combination of three terms

$$EC \approx EC_{\infty}^{SF} + \Delta_{\infty}^{MF} + \Delta^{GA}$$

where EC_{∞}^{SF} denotes the EC for systematic loss in a "properly chosen single-factor model",[26] Δ_{∞}^{MF} is a correction accounting for the effect of all systematic factors, and Δ^{GA} is a correction

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[24] Wilde (2001) takes an alternative, direct approach.

[25] Gordy (2003, 2003b) presented this approach first and was then refined in Wilde (2001) Martin and Wilde (2002).

[26] This is different form the single-factor EC as defined in equation (5).

accounting for idiosyncratic risk (similar to the standard *GA* in Gordy 2003 and Wilde 2001). To the degree that the Pykhtin model and the *DF* model are correct, we have

$$DF \cdot EC^{SF} \approx EC_{\infty}^{SF} + \Delta_{\infty}^{MF}$$

In order to provide a single framework to understand sector/geographic and name concentrations, in this section, we have taken here a simple top-down approach for the *GA* in the context of a model with implied parameters which can be fitted to the "true" EC computed from the MC simulation model, or in practice also using the Pykhtin model.

Example

We present a stylized example to illustrate the behaviour of the model, the computation of implied parameters, and its application to compute sensitivities, stress testing and marginal capital of new positions.

Consider the credit portfolio with four sectors given in Table 5. The first two sectors have a *PD* of 1% and exposure of 25; the other two sectors have a lower *PD* (0.5%). For simplicity, assume a 100% *LGD*. The third and fourth columns show the expected losses (*EL*) in dollar values and as percent of total *EL*. The following two columns give the single-factor (SF), stand-alone, capital (total and percent). The last column shows the intra-sector correlation (ρ) for each sector. The portfolio total exposure is 100, the *EL* is 75bps and the single-factor portfolio capital is 9.7%. The *CDI* of almost 33% implies roughly three effective sectors.

Portfolio	EAD	PD	EL	EL %	Capital (Single Factor)	Capital % (Single Factor)	Rho
P1	25	1.0%	0.25	33.3%	3.4	35.3%	**20.1%**
P2	25	1.0%	0.25	33.3%	2.1	21.5%	**12.4%**
P3	40	0.5%	0.20	26.7%	3.8	39.6%	**21.9%**
P4	10	0.5%	0.05	6.7%	0.4	3.7%	**8.6%**
Total	**100**		0.75	100.0%	**9.7**	100.0%	
					CDI	**32.9%**	

Table 5. Four-sector portfolio: characteristics and stand-alone capital

We observe the effect of various credit parameters by comparing the contributions to total exposure, *EL* and SF capital. The differences in exposure and *EL* contributions can be explained by the interaction of the exposures (and *LGDs*) with the *PDs.* Intra-sector correlations explain the differences between *EL* and capital contributions. For example, the fourth sector represents one

tenth of the total exposure, almost 7% of *EL*, but less than 4% of SF capital. This indicates that it is a low *PD* sector with a lower than average intra-sector correlation. In contrast, the third sector constitutes 40% of total exposure, 27% of *EL* and about 40% of SF capital. This sector's low *PD* reduces its *EL* contribution, but its higher correlation (22%) increases its share of SF capital. The first sector's high capital contribution is explained by both high *PD* and intra-sector correlation.

To illustrate the computation of implied parameters, assume that the bank has a multi-factor MC-based credit portfolio model (already parameterized), which gives a portfolio EC of 7.3% of the total exposure. From this "true" EC we imply the *DF* and average inter-sector factor correlation. The results are summarized in Table 6. The *DF* is 75.5% (7.3 = 0.755 x 9.7) and the implied the average correlation is $\overline{\beta} = 54.9\%$.

Portfolio	Exposure	Capital (Single-Factor)	Rho	EC % (Flat Beta=54.6%)	EC %	Implied Qk
P1	25	35.3%	20.1%	36.1%	31.9%	**45.7%**
P2	25	21.5%	12.4%	19.0%	17.2%	**49.7%**
P3	40	39.6%	21.9%	42.3%	47.5%	**65.6%**
P4	10	3.7%	8.6%	2.6%	3.4%	**66.8%**
Total	100					

Capital (Single-Factor)	CDI	DF	EC	Implied Average Beta
9.7	32.9%	**75.6%**	**7.3**	**54.9%**

Table 6. Multi-factor capital and implied inter-sector correlations

The fifth column of Table 10 gives the EC contributions assuming that all sector factor correlations are equal to the average of 54.9%. In this case, every sector factor is equally correlated with the overall portfolio, and the only difference stems from the size component of the sector diversification factors ΔDF_k^{size}. These contributions are close but not equal to the SF capital contributions. The decomposition of the sector diversification factor for a flat correlation is given on the left side of Table 7. Compared to the single-factor case, the size component of the sector diversification factor increases contributions for the two biggest sectors (P1 and P3) and decreases them for the two small ones (P2 and P4). While the overall diversification factor is 75.6%, the marginal sector diversification factors range from 53% (P4) to 81% (P3).

	Flat Sector Factor Correlation (Average)				Implied Sector Factor Correlations			
Portfolio	DF_k	Portfolio Diversification	Sector Size	Sector Correlation	DF_k	Portfolio Diversification	Sector Size	Sector Correlation
P1	77.5%	75.6%	1.8%	0%	68.4%	75.6%	1.8%	-9.1%
P2	66.9%	75.6%	-8.7%	0%	60.7%	75.6%	-8.7%	-6.2%
P3	80.8%	75.6%	5.2%	0%	90.6%	75.6%	5.2%	9.8%
P4	53.3%	75.6%	-22.3%	0%	70.7%	75.6%	-22.3%	17.4%

Table 7. Decomposition of marginal sector diversification factors.

Next, assume that the MC-based portfolio model calculates the "true" marginal EC contributions given in the sixth column in Table 6. Note that these are different from those obtained assuming a flat inter-sector correlation (fifth column in Table 10)). The implied $\overline{Q}_k$'s for each sector (as well as an implied correlation structure for the sector factors) are calculated from these true capital contributions (last columns of Table 6).

For the first two sectors, the "true" EC contributions are lower than those with flat correlations, which results in lower than average implied correlations of their sector factors to the rest of the portfolio. The right side of Table 7 gives the decomposition of the sector diversification factors. The first two sectors have negative sector correlation diversification components (last column). The opposite is true for P3 and P4 (higher than average implied correlations and positive correlation component in the marginal sector *DF*).

The fitted *DF* model provides analytical sensitivities to all the parameters and can now be used to compute, instantaneously, stress tests or capital contribution of new loans, while providing an explanation of the sources of risk and diversification. In what follows, we give several examples.

First, the derivatives of *DF* and *EC* with respect to the average sector correlation are

$$\partial DF \big/ \partial \overline{\beta} = 0.5\,, \quad \partial EC^{mf} \big/ \partial \overline{\beta} = 4.9$$

Thus, a 5% increase in average inter-sector correlation (from today's level at 55% to 60%) results in an increase in capital of 0.25 (from 7.3 to 7.55). Figure 11 further presents a stress test of average inter-sector correlation. It shows that (at this level of *CDI*), EC behaves linearly with average inter-sector correlation.

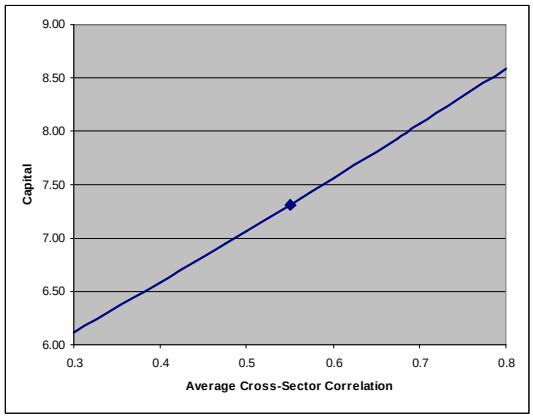

Figure 11. Average correlation stress test.

Consider now a new loan in Sector 1 with a *PD*=1%. A new small exposure to Sector 1 roughly adds 0.136 single-factor capital per unit of exposure (the sector's SF capital of 3.4 divided by its exposure of 25). In contrast, given diversification, it only results in additional marginal EC of 0.093 per unit of exposure (the product of it's SF capital and the marginal *DF* of 68.4%). Although this is sector with a high PD and ρ, the benefit of diversification is greater, since it is the least correlated sector (the smallest Q_k).

We can further investigate the impact on EC of new non-marginal portfolios of transactions in Sector 1, as the effect of other sector parameters. Figure 12 presents standardized stress tests of exposure, *PD* and intra-sector correlation for Sector 1. In each case, the parameters lie between half and double their current values. The results are expressed as percent changes of the current value (for both, the portfolio SF capital and EC).

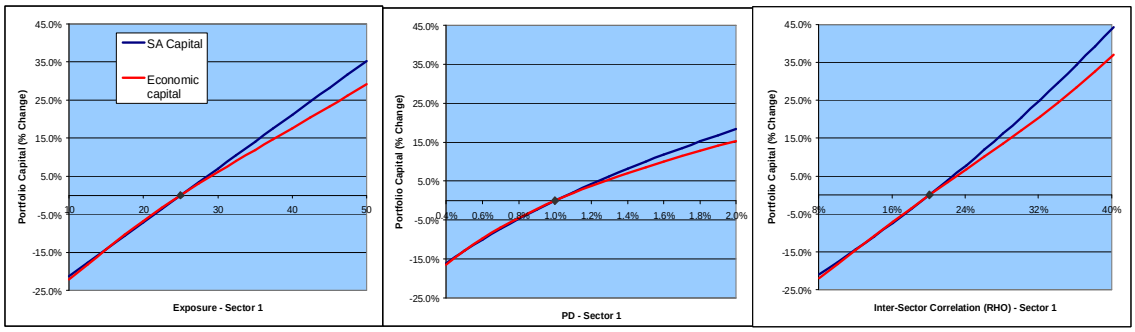

Figure 12. Parameter stress tests for Sector 1.

While SF capital is linear in exposures, EC shows convexity when adding bigger positions. Stress test of *PD*s and correlation provide useful guidance on model risk or the effects of changing

economic conditions. For example, conservative capital estimates assuming a 50% higher correlation level (from 20.1% to 30.2%) for Sector 1 result in additional 20.5% SF capital for the portfolio and 17% higher EC. In all cases, the percent capital impact is smaller for EC (this is true for this sector, but not necessarily so for all sectors).

Finally, Figure 13 presents a stress test of EC against all the sector *PD*s. EC most sensitive to Sector 3 (the largest sector with the lowest diversification contribution). In contrast, a change of *PD* in Sector 4 (which is the smallest) has a very small impact.

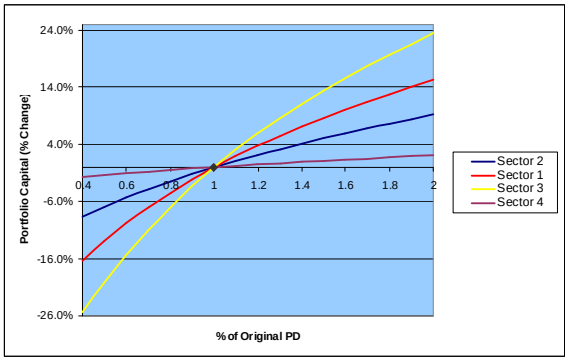

Figure 13. Sector *PD* stress test.

5. Concluding Remarks

We present a simple adjustment to the single-factor credit capital model, which recognizes the diversification obtained from a multi-factor setting. In contrast to MC methods, there are benefits for seeking analytical approximations both for regulatory purposes as well as for credit portfolio management. As a risk management tool, the model can be used to understand concentration risk, capital allocation and sensitivities, as well as to perform stress testing and compute "real-time" marginal risk for new deals or portfolios.

The model is based on the concept of a *diversification factor*. We estimate the diversification factor for a family of multi-factor models, and show that it can be expressed as a function of two parameters that broadly capture the size concentration and the average cross-sector correlation. The model further supports an intuitive capital allocation methodology. For this purpose we define *marginal diversification factors* at the sub-portfolio (or obligor) level, which account for their diversification contributions to the portfolio.

The estimation of the diversification factor surface requires substantial numerical work. Once estimated, however, it can then be expressed parametrically or tabulated. This results in a practical, simple and fast method that can be used for risk management including stress testing and pre-deal analytics. In addition, the model can be calibrated to a MC-based credit portfolio model on a periodic basis (e.g. monthly or even daily) to adjust for changing market conditions and portfolio composition. The model can then be used during the day to support decision making in real time, origination and trading.

We believe the diversification factor has potential to be applied to extend the Basel II regulatory framework to a general multi-factor setting, thus allowing for more accurate modelling of diversification within portfolios across various asset classes, sectors and regions. However, a few remarks are appropriate with respect to its calibration together with the regulatory parameters from Basel II. While we have used in Section 3 the Basel formulae for wholesale exposures, we do not wish to imply that, as presented, the calibration exercises are generally appropriate for regulatory rules or that the economic capital from a multi-factor model should always be smaller than the Basel II capital. One can argue that, if the sample used for calibrating a single-factor model such as in Basel II already covers the sectors in the portfolio, the asset correlations ρ_k already account, to a large extent, for cross-sector diversification (see also e.g. Lopez 2004). To the degree that the original parameter calibration accounts for cross sector diversification, some scaling (up) for intra-sector correlations or (down) the diversification factor is required, in order to not incur in double counting.

Finally, there are several enhancements of the model, which can be addressed in future research:

- The *DF* presented only covers systemic credit risk (as does the Basel II model) and it was extended in Section 4 to cover granular portfolios in a simple way. Its current strength is on capturing sector and geographical concentrations. A useful extension of the model would further refine the coverage of name concentrations in a more rigorous way.

- There is potential for improving the parameterization of the model. More parameters can be added or perhaps one can search for parameters that result in a better or more general fit. In our opinion, this should not be done at the expense of too much complexity or of loosing the intuitive interpretation of its parameters, results and capital allocation.

- We have formulated how risk concentrations work within this model. Further work is needed to explore their mathematical behaviour, their role in model calibration and further application in practice.

- Perhaps the most obvious limitation of the model today is its reliance on costly numerical estimation. Ideally, we would like also a closed form approximation for the *DF* that is accurate and perhaps does not rely as much on numerical calibration. As such, for example, the known solution for Normal distributions may provide useful insights into the more general problem.

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Appendix. Derivation of Marginal Diversification Factors

We briefly outline the derivation of the expressions for the *marginal sector diversification factors* DF_k in equation (16). The first step is to note that the EC in equation (13), EC^{mf}, is a homogeneous function of degree one in the EC_k's. A function $f(x_1,...,x_n)$ is said to be (positively) homogeneous of degree *a* if

$$f(kx_1,...,kx_n) = k^a f(x_1,...,x_n), \text{ for } k \text{ a (positive) real number}$$

To see that EC^{mf} is homogeneous of degree one, we write equation (13) as

$$EC^{mf} = DF(CDI, \overline{\beta}) \cdot \sum_{k=1}^{K} EC_k$$

and observe that *DF* only depends on *CDI* and $\overline{\beta}$, which are both homogenous functions of degree zero. Homogeneous functions of degree one satisfy Euler's theorem:

$$f\left(x_1,...,x_n\right)=\frac{\partial f}{\partial x_1}x_1+...+\frac{\partial f}{\partial x_n}x_n$$

This leads to the additive marginal capital decomposition (14) with

$$DF_k=\frac{\partial EC^{mf}}{\partial EC_k} \quad , \quad k=1,...,K \tag{15}$$

To obtain expression (16), we explicitly take the partial derivatives on the right side of expression (15), by applying the chain rule, as follows:

$$\frac{\partial EC^{mf}}{\partial EC_k}=DF\left(CDI,\overline{\beta}\right)\cdot\frac{\partial EC^{sf}}{\partial EC_k}+EC^{sf}\cdot\frac{\partial DF\left(CDI,\overline{\beta}\right)}{\partial EC_k}$$

$$=DF+EC^{sf}\cdot\frac{\partial DF}{\partial CDI}\cdot\frac{\partial CDI}{\partial EC_k}+EC^{sf}\cdot\frac{\partial DF}{\partial\overline{\beta}}\cdot\frac{\partial\overline{\beta}}{\partial EC_k}$$

Equation (16) is finally obtained by taking the derivatives of the *CDI* (equation 7) and of the average correlation (equations 11 and 12) with respect to the stand-alone capital for the sector and arranging terms.